THE THEORY OF UNCONVENTIONAL MONETARY POLICY

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Abstract. This paper is about the effectiveness of qualitative easing, a form of unconventional monetary policy that changes the risk composition of the central bank balance sheet with the goal of stabilizing economic activity. We construct a general equilibrium model where agents have rational expectations and there is a complete set of financial securities, but where some agents are unable to participate in financial markets. We show that a change in the risk composition of the central bank's balance sheet will change equilibrium asset prices and we prove that, in our model, a policy in which the central bank stabilizes non-fundamental fluctuations in the stock market is Pareto improving and self-financing.

Central banks throughout the world have recently engaged in two kinds of unconventional monetary policies: *quantitative easing* (QE), which is "an increase in the size of the balance sheet of the central bank through an increase in its monetary liabilities", and *qualitative easing* (QualE) which is "a shift in the composition of the assets of the central bank towards less liquid and riskier assets, holding constant the size of the balance sheet."[1]

Because qualitative easing is conducted by the central bank, it is often classified as a monetary policy. But because it adds risk to the public balance sheet that is ultimately borne by the taxpayer, QualE is better thought of as a fiscal or quasi-fiscal policy (Buiter, 2010). This distinction is important because, in order to be effective, QualE necessarily redistributes resources from one group of agents to another.

In theoretical papers that study the effectiveness of QualE, researchers often assume that financial markets are complete and that there is complete participation in the financial markets. When these two conditions hold, a change in the risk composition of the central bank's balance sheet has no effect on asset prices.

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An earlier incarnation of this paper with the title "Qualitative Easing: How it Works and Why it Matters" (Farmer, 2012d) appeared as an NBER and a CEPR working paper. The current version was written after extensive discussions with Pawel Zabczyk during, and following, Farmer's visit, as Senior Houblon-Norman Fellow at the Bank of England in 2013. In contrast to the earlier paper, here we introduce money. Farmer thanks the Center for Central Bank Studies at the Bank of England for their hospitality. We thank participants at a seminar at the National Institute for Economic and Social Research for their comments. We have also benefited from discussions with Andy Haldane, Minouche Shafik and Martin Weale. The views expressed in this paper are those of the authors, and not necessarily those of the Bank of England.

[1]The quote is from Willem Buiter (2008) who proposed this very useful taxonomy in a piece on his 'Maverecon' Financial Times blog. Farmer (2013) used this distinction to argue that the Bank should actively stabilize the asset markets. This paper provides the theory that explains why Qualitative Easing is effective. Earlier working papers that explain why financial markets may be Pareto Inefficient include Farmer (2002b,a, 2012c, 2014, 2015) and Farmer et al. (2012).

For example, in an influential piece that was presented at the 2012 Jackson Hole Conference, Michael Woodford (2012) made the claim that QualE is unlikely to be effective and, to the extent that it does stimulate economic activity, that stimulus must come through the impact of QualE on the expectations of financial market participants of future Fed policy actions.[2] In contrast, Joseph Gagnon (2012) has argued that qualitative easing works through the *portfolio balance effect*, a term attributed to James Tobin (1963; 1969) who assumed that private agents' asset demands are functions of relative asset prices, much as the demands for commodities depend on relative goods prices.[3]

QualE is ineffective when participation in assets markets is complete because market participants are able to undo the effects of a portfolio shift by the central bank through private trades in securities. As a consequence, QualE has no effect on the distribution of resources, either between borrowers and lenders in the current financial markets, or between current market participants and those yet to be born. We will argue here, that the assumption of complete participation is not a good characterization of real world financial markets and that QualE is effective because it redistributes resources across states of nature for people who are unable to participate in financial markets that open before they are born.

We make the case for the effectiveness of qualitative easing by constructing a simple general equilibrium model where agents are rational and have rational expectations and where the financial markets are complete. Our model has two important features. First, the people in our model use money as a medium of exchange. This feature ensures that, in the absence of uncertainty, the model possesses multiple equilibria. Second, some people in our model are not present in the financial markets. That property is a feature of all real world economies where it follows from the fact that people have finite lives.

In this environment, we show that 1) a central bank that takes risk onto its balance sheet can increase welfare and 2) the optimal intervention restores efficiency and is self-financing. When all uncertainty is non-fundamental, the optimal policy is for the central bank to stabilize the stock market so that the return to the stock market is equal in every state to the return on a one-period real government bond. In the presence of fundamental shocks, the government intervention eliminates non-fundamental volatility.

1. A Simple Model

Our model is highly stylized. There are two periods, three types of people and two public agents; a central bank and a treasury. We refer to type 1 and 2 people as workers and to type 3 people as entrepreneurs.

Workers are alive in both periods and they are each endowed, in period 2, with one unit of leisure. Entrepreneurs are alive only in period 2. They are endowed with a technology for producing a unique consumption good in period 2. The fact that we call these people 'entrepreneurs' is not important to the economic arguments that we will make. In a more

[2]Citing papers by Krugman (1998) and Eggertsson and Woodford (2002) where the case is made explicitly, Woodford (2012) argues that this so-called portfolio balance view is invalid, and, if central bank asset purchases are to be effective, their effectiveness must rely on their ability to alter the public's expectations of future central bank policies.

[3]Examples of recent empirical papers that find a significant effect of Fed asset purchases on asset prices include Vissing Jorgensen and Krishnamurthy (2011); Gagnon et al. (2011); D'Amico and King (2010); Neely (2010); Li and Wei (2012), and Hamilton and Wu (2012).

complicated model with multiple periods and long lives there would be people of all types present in all periods.

In period 1, workers trade in asset markets with each other, with the central bank, and with the treasury. Production and consumption takes place only in period 2. There is a paper asset called money, that is useful in exchange. To capture the importance of money we assume that the real value of cash balances is an argument of workers' utility functions.[4]

2. The Budget Constraints of a Worker

Workers face the following budget constraint in period 1,

$$M_i + QA_i - TR_i = 0,$$

where, TR_i is a nominal transfer to workers by the treasury, M_i is accumulation of money, A_i, is accumulation of one-period dollar denominated interest bearing assets and Q is the period 1 price of a dollar-denominated pure discount bond. The subscript $i \in \{1, 2\}$ indexes workers. In period 2, workers face the constraint,

$$pc_i + w(1 - n_i) \leq w + M_i + A_i - \mathcal{T}_i.$$

Here, w is the money wage, p is the price of commodities, n_i is labor supply, c_i is consumption and $\mathcal{T}_i$ is a lump-sum tax obligation. Putting together the budget constraints of workers for periods 1 and 2, and rearranging terms, leads to the life-cycle constraint,

$$pc_i + w(1 - n_i) + rM_i \leq \mathcal{W}_i, \tag{1}$$

where

$$\mathcal{W}_i \equiv w + \frac{TR_i}{Q} - \mathcal{T}_i, \tag{2}$$

is the dollar value, at date 2, of a worker's wealth and

$$r \equiv \frac{1 - Q}{Q}, \tag{3}$$

is the money interest rate.

3. The Budget Constraints of the Treasury and the Central Bank

The treasury makes dollar denominated lump-sum transfers in period 1, to types 1 and 2, paid for by issuing dollar denominated debt,

$$QB = \sum_i TR_i \equiv TR. \tag{4}$$

The left side of this equation is the value of debt floated by the treasury in the asset markets and the right side is the value of the dollar denominated transfers to the private sector. The value of debt is broken into the face value of dollar denominated pure discount bonds, denoted by B, and their price at date 1, denoted by Q. We assume that the treasury picks a value of B and that it disburses the revenues $TR = QB$ to workers in the form of a transfer. B is our fiscal policy variable and we refer to the choice of B as a *fiscal policy.*

Asset market trades in period 1 involve workers, the treasury and the central bank. An amount A_{CB} of treasury debt is purchased by the central bank and the remaining portions A_1

[4]Money in the utility function has a distinguished history dating back to Patinkin (1956).

and A_2 are purchased by workers. This leads to the following asset market clearing equation for government debt,

$$QB = QA_{CB} + QA_1 + QA_2.$$

The portion of the debt purchased by the central bank is equal to the money supply and is held by private agents as checking accounts at the central bank,

$$QA_{CB} = M. \tag{5}$$

M is our monetary policy variable and we refer to the choice of M as a *monetary policy*.

Because the bank does not pay interest on its liabilities, the creation of money generates equity for the central bank equal to the present value of the bank's seigniorage revenues,

$$E_{CB} = Q\mathcal{S},$$

where $\mathcal{S}$ is defined as,

$$\mathcal{S} \equiv (A_{CB} - M) = rM. \tag{6}$$

Table 1 represents the balance sheet of the central bank in period 1.

Assets	Liabilities
QA_{CB}	M
$Q\mathcal{S}$	
	E_{CB}

TABLE 1. The Central Bank Balance Sheet

At date 2, the seigniorage revenues from money creation are repaid to the treasury. The treasury must repay its debt by raising taxes $\mathcal{T}$ on the private sector, or from seigniorage revenues, $\mathcal{S}$,

$$B = \mathcal{T} + \mathcal{S}. \tag{7}$$

Replacing B in Equation (4) from Equation (7) leads to the intertemporal government budget constraint,

$$Q(\mathcal{T} + \mathcal{S}) = TR.$$

This equation clarifies that the dollar value of the transfer to the workers in period 1 is equal to the present value of tax revenues plus the present value of seigniorage revenue.

4. The Equal Treatment Assumption

We assume that people alive in each period are treated equally, and thus each worker receives half the transfer

$$TR_1 = TR_2 = \frac{TR}{2}, \tag{8}$$

and workers and the entrepreneurs each pay a third of the tax burden,

$$\mathcal{T}_1 = \mathcal{T}_2 = \mathcal{T}_3 = \frac{\mathcal{T}}{3}. \tag{9}$$

We can then express wealth $\mathcal{W}_i$ in equation (2) as

$$\mathcal{W} \equiv w + \left(\frac{B}{6} + \frac{rM}{3}\right), \tag{10}$$

where the equal treatment assumptions imply that $\mathcal{W}$ is identical for both workers, allowing us to drop the subscript i.[5] This expression makes it clear that the value of the transfer to workers depends on both fiscal policy, represented by B, and monetary policy, represented by M.

Fiscal policy in our model is a pure transfer from one generation to another. All government expenditures in period 1 are in the form of transfer payments to workers. Those transfers are repaid in period 2 by taxes levied on both workers and entrepreneurs and from seigniorage revenues. Because entrepreneurs are not present in period 1, they do not benefit from transfers. They do, however, incur part of the cost of paying for those transfers.[6]

5. Workers

Workers have logarithmic preferences defined over consumption, leisure and the real value of money balances in period 2 with weights λ_i on consumption, μ_i on leisure and γ_i, on real money balances defined in wage units,[7]

$$U_i = \lambda_i \log c_i + \mu_i \log(1 - n_i) + \gamma_i \log\left(\frac{M_i}{w}\right),$$

where

$$\lambda_i + \mu_i + \gamma_i = 1.$$

We assume that workers differ in their preference weights and we use the subscript i to index type. Workers maximize utility subject to the lifecycle constraint, Inequality (1). The solution to this problem, given the assumption of logarithmic preferences, is for the workers' expenditure shares on leisure, consumption and money to equal the respective coefficients in the utility function times wealth, $\mathcal{W}$, that is,

$$w(1 - n_i) = \mu_i \mathcal{W}, \tag{11}$$

$$pc_i = \lambda_i \mathcal{W}, \tag{12}$$

$$rM_i = \gamma_i \mathcal{W}. \tag{13}$$

Rearranging terms in Equation (11), we obtain the following expression for the labor supply function of type i,

$$n_i = 1 - \mu_i \frac{\mathcal{W}}{w}. \tag{14}$$

[5] In words, Equation (10) says that the period 2 money value of the wealth of a young worker is equal to the money value of his leisure endowment plus 1/6 of the period 2 value of his transfer plus 1/3 of the government's seigniorage revenue from money creation. The fraction 1/6 appears because workers receive half of the government transfer but only have to repay a third of it (1/6 = 1/2 − 1/3). The fraction 1/3 of seigniorage revenue follows from the fact that, for a given transfer, additional seigniorage revenues reduce the tax burden on all three types. More generally, the wealth effect of a transfer policy will depend on the population growth rate and the period length.

[6] It would be relatively simple to extend our model to allow for government purchases and for trade in goods in period 1. We have not pursued that extension because our goal in this paper is to focus on the role of trade in the asset markets. In our view, central bank asset trades have non-trivial effects because they generate fiscal transfers between generations that cannot be undone by private markets. We show subsequently that these trades can sometimes be Pareto improving.

[7] The assumption of logarithmic utility is not important for our results. All the qualitative properties of our model can be derived for the case of general concave utilities.

6. Entrepreneurs

Entrepreneurs do not participate in the asset markets since, by assumption, they are born in period 2. Each entrepreneur owns a decreasing returns-to-scale technology,

$$y = n^{\alpha},$$

that transforms labor into output. Entrepreneurs receive real profits, Π, defined as,

$$\Pi \equiv n^{\alpha} - \frac{w}{p} n,$$

and their consumption is subject to the constraint

$$c_3 \leq \Pi - \frac{\mathcal{T}_3}{p}. \tag{15}$$

Using the equal treatment assumption, Equation (9), and the consolidated government budget constraint, Equation (7), we may write taxes, $\mathcal{T}_3$ as,

$$\mathcal{T}_3 = \frac{B - rM}{3}.$$

Entrepreneurs choose c_3 and n to solve the problem

$$\max_{\{c_3, n\}} U_3 = \log(c_3),$$

such that $c_3 \leq \Pi - \frac{\mathcal{T}_3}{p}$. The solution is characterized by the labor demand function,

$$n = \left(\frac{1}{\alpha} \frac{w}{p} \right)^{\frac{1}{\alpha - 1}}, \tag{16}$$

the output supply function,

$$y = \left(\frac{1}{\alpha} \frac{w}{p} \right)^{\frac{\alpha}{\alpha - 1}}, \tag{17}$$

and the entrepreneur's consumption demand function,

$$c_3 = (1 - \alpha) \left(\frac{1}{\alpha} \frac{w}{p} \right)^{\frac{\alpha}{\alpha - 1}} - \frac{(B - rM)}{3p}. \tag{18}$$

We have assumed that entrepreneurs have logarithmic utility. That assumption is not important for the construction of a perfect foresight equilibrium. As long as their utility is increasing in consumption, entrepreneurs will maximize profit Π. But the assumption that preferences are logarithmic implies that entrepreneurs and workers are risk averse and this property will affect outcomes when we discuss equilibria under uncertainty in Section 12. Because of risk aversion, equilibria where non-fundamental shocks influence the allocation of goods across states are Pareto inefficient.

7. Perfect Foresight Equilibria

There are three goods: labor, consumption, and money, and three dollar denominated prices: p, w and r. In this section, we characterize equilibria as the solutions to the excess demand functions for these three goods. To simplify this characterization, we first define

$$\lambda \equiv \sum_{i \in \{1,2\}} \lambda_i, \qquad \mu \equiv \sum_{i \in \{1,2\}} \mu_i, \qquad \gamma \equiv \sum_{i \in \{1,2\}} \gamma_i. \tag{19}$$

Using these definitions, the labor market clearing equation is,

$$\overbrace{\left(\frac{1}{\alpha}\frac{w}{p}\right)^{\frac{1}{\alpha-1}}}^{\text{Labor Demand}} = \overbrace{2-\mu\frac{\mathcal{W}}{w}}^{\text{Labor Supply}} , \tag{20}$$

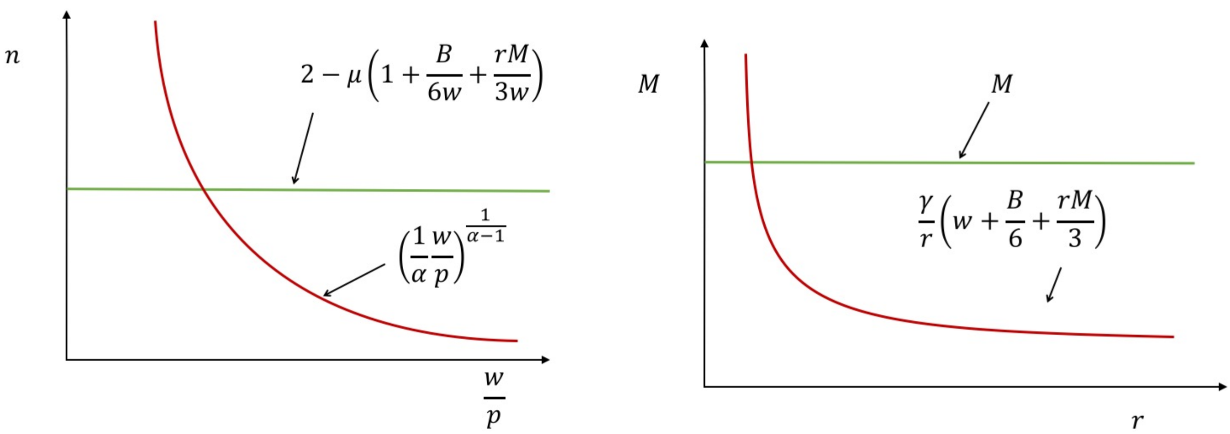
the goods market clearing equation is,

$$\overbrace{\left(\frac{1}{\alpha}\frac{w}{p}\right)^{\frac{\alpha}{\alpha-1}}}^{\text{Output}} = \overbrace{(1-\alpha)\left(\frac{1}{\alpha}\frac{w}{p}\right)^{\frac{\alpha}{\alpha-1}} - \frac{(B-rM)}{3p}}^{\text{Entrepreneur's Consumption}} + \overbrace{\frac{w}{p}\lambda\frac{\mathcal{W}}{w}}^{\text{Workers' Consumption}} , \tag{21}$$

and the money market clearing equation is,

$$\overbrace{\gamma\frac{\mathcal{W}}{r}}^{\text{Money Demand}} = \overbrace{M}^{\text{Money Supply}} , \tag{22}$$

where $\mathcal{W}$ is defined in Equation (10).



(A) Labor Market Clearing. (B) Money Market Clearing.

Figure 1. The Demand and Supply of Labor and Money.

Figure 1(A) plots the demand and supply for labor as functions of the real wage and the money interest rate. The real wage appears on the horizontal axis and the quantities of labor demanded and supplied are on the vertical axis. The downward sloping curve is the labor demand function; this is the left-hand-side of Equation (20). The horizontal line is the labor supply function; this is the right-hand-side of Equation (20). The function is horizontal because, when people have logarithmic preferences, the income and substitution effects exactly balance each other.

Figure 1(B) plots the demand and supply of money as functions of the money wage, w, and the money interest rate r. The horizontal line is the money supply. The downward sloping curve is the demand for money as a function of r.

Definition 1. *An equilibrium is a monetary policy and a fiscal policy* $\{M, B\}$, *an allocation* $\{\{n_i, M_i\}_{i\in 1,2}, \{c_i\}_{i=1,2,3}\ , y, n\}$ *and a set of prices* $\{p, w, r\}$ *that satisfies non-negativity, budget balance and optimality. An equilibrium price system is a non-negative triple* $\{p, w, r\}$ *such that equations (20), (21) and (22) hold.*

Proposition 1 establishes that there is a continuum of equilibria and characterizes them in closed form.

Proposition 1. *Let $\{M, B\} \geq 0$ characterize monetary and fiscal policy, and let $w > 0$ satisfy the feasibility conditions,*

$$w \geq \frac{\mu_i B}{4(1+\lambda+\mu) - 6\mu_i}, \quad i \in \{1, 2\} \quad \textit{and} \quad w \geq \frac{2 - \mu + \lambda(2 - 3\alpha)}{\mu + \alpha\lambda} \frac{B}{2}. \tag{23}$$

The equilibrium level of nominal wealth, the interest rate, the real wage and are given by,

$$\mathcal{W} = \frac{6w + B}{2(1+\lambda+\mu)}, \quad r = \frac{\gamma}{M}\mathcal{W}, \quad \frac{w}{p} = \alpha\left(2 - \frac{\mu\mathcal{W}}{w}\right)^{\alpha-1}. \tag{24}$$

The equilibrium values of $\{\{n_i, M_i\}_{i\in 1,2}, \{c_i\}_{i=1,2,3}\ , y, n\}$ are determined by equations (11) – (13) and (16) – (18) respectively. □

See Appendix A for a proof of this proposition.

The fact that there is a continuum of equilibria follows from the fact that transfers are defined in nominal units but they have real effects on the allocation of resources between workers and entrepreneurs. Although we have proven this in a two period model, the result is more general. Every general equilibrium model of money contains multiple equilibria since there is always at least one equilibrium where money has value and a second equilibrium where it does not (Hahn, 1965). Generically, models in this class also contain a continuum of non-stationary equilibria for which the inflation rate converges to the rate associated with one of the stationary equilibria.

8. Beliefs and Equilibrium Selection

Because dynamic monetary models always have multiple equilibria we must take a stand on what selects an equilibrium. Multiplicity occurs in dynamic models because there is more than one possible future price. A decision maker, placed in an environment where many different things can happen, must take an action. Following Farmer (2012b), we select what will happen by modeling the way that people form beliefs as a new primitive of the model with the same methodological status as preferences and technology.

In the 1960s, it was common to distinguish the future value of a price, we will call this P, from the expectation of that price, we will call this P^E. Because P and P^E were modeled as separate objects the researcher was obliged to introduce a new equation to explain how P^E is determined. Following the work of Friedman (1957), macroeconomists often assumed that expectations are formed adaptively. Under the adaptive expectations hypothesis,

$$P_{t+1}^E = \psi P_t^E + (1-\psi)P_t,$$

where ψ is a parameter that determines the speed of adjustment of the belief about the future price to its actual value and the subscript t denotes time period.

With the introduction of rational expectations by Robert Lucas (1972), the way that economists model expectations changed. Lucas argued that the world is uncertain and that the people who inhabit our models would be expected to adapt to uncertainty. He suggested that prices fluctuate because of random shocks to the fundamentals of the economy. If the economy enters some state, captured by the random variable X, the price we observe will be a function, $P(X)$. If X is stationary, every time the world enters state X the people who inhabit our model would expect to observe the same price, $P(X)$. Eventually, they will learn the mapping from X to $P(X)$ and they will form their expectations $P^E(X)$ using the equilibrium price function $P(X)$.

That argument was widely accepted and for the past forty years, almost all macroeconomic models have adopted the rational expectations assumption. The argument is persuasive; but it relies on the assumption that the rational expectations equilibrium is unique. In monetary models, the uniqueness property is almost always violated. In models with multiple rational expectations equilibria we cannot dispense with an equation that explains how expectations are formed. Farmer (2002c) referred to that equation as the belief function and we will adopt that same terminology here.

The belief function, $\varphi(\cdot)$, is a mapping from present and past observable variables to a probability measure over future prices. For example, the people in our model might believe that,

$$w^E = \varphi(B, M, \varepsilon),$$

where w^E is the anticipated money wage next period and ε is a random variable with probability distribution,

$$\varepsilon = \begin{cases} H & \text{with probability} \quad \pi_H \\ L & \text{with probability} \quad \pi_L. \end{cases}$$

Here, ε captures the possibility that people may over-react or under-react to fundamental uncertainty as well as the possibility that they may react to purely non-fundamental shocks. If expectations are rational, the model will be closed by the equation,

$$w^E(\varepsilon) = w(\varepsilon), \quad \varepsilon \in \{H, L\}.$$

In words, the realization of the money wage, $w^E(\varepsilon)$, that people expect to occur if the random variable ε is realized, will coincide with $w(\varepsilon)$, the value of w that actually occurs.

Consider the following simple example of a belief function. People believe that w may take one of two values,

$$\varphi(M, \varepsilon) = M + \varepsilon \equiv \begin{cases} M + H \equiv w\,(H) & \text{with probability} \quad \pi_H \\ M + L \equiv w\,(L) & \text{with probability} \quad \pi_L. \end{cases} \tag{25}$$

We have included the policy variable M in the belief function to capture the idea that beliefs depend on observable variables. In this example, beliefs are independent of B but they do depend on M. We have included the random variable ε in the belief function to capture the idea that non-fundamental uncertainty may matter simply because people believe that it will. We will demonstrate in the next section that non-fundamental beliefs cause allocations to fluctuate even when there is a complete set of insurance markets.[8]

9. Budget Constraints Under Uncertainty

In this section we introduce uncertainty and we assume that people form beliefs using Equation (25). If $\varepsilon = H$, the realization of the money wage is equal to $w\,(H)$ and if $\varepsilon = L$

[8]Our example, where money is the only fundamental that affects beliefs, is very special. More generally, beliefs about the future wage might depend on current and past wages, or on current and past output or employment. For an example of a complete model closed with a belief function see Farmer (2012a).

the realization of the money wage is equal to $w(L)$. These rational expectations influence the trades that workers make in period 1.

To model complete insurance markets we assume the existence of a pair of Arrow securities (Arrow, 1964), one for each state. The H security pays one dollar if and only if state H occurs and the L security pays one dollar if and only if state L occurs. The H security costs $Q(H)$ dollars in period 1 and the L security costs $Q(L)$ dollars.

In period 1, workers trade in the asset markets. Each worker receives a transfer TR_i that he may use to acquire money M_i or Arrow securities $A_i(\varepsilon)$,

$$\sum_{\varepsilon\in\{H,L\}} Q(\varepsilon) A_i(\varepsilon) + M_i \leq TR_i.$$

In period 2 state ε, he faces the constraint,

$$p(\varepsilon) c_i(\varepsilon) + w(\varepsilon)(1 - n_i(\varepsilon)) \leq w(\varepsilon) + A_i(\varepsilon) + M_i - \mathcal{T}_i.$$

Substituting for $A_i(\varepsilon)$ from the period 2 constraint into the first period gives the following lifecycle budget constraint,

$$\sum_{\varepsilon\in\{H,L\}} Q(\varepsilon)\left[p(\varepsilon) c_i(\varepsilon) + w(\varepsilon)(1 - n_i(\varepsilon)) - w(\varepsilon) - M_i + \mathcal{T}_i\right] + M_i \leq TR_i.$$

Defining state prices as,[9]

$$\tilde{w}(\varepsilon) \equiv \frac{Q(\varepsilon)}{Q}\frac{w(\varepsilon)}{\pi_\varepsilon} \quad \text{and} \quad \tilde{p}(\varepsilon) \equiv \frac{Q(\varepsilon)}{Q}\frac{p(\varepsilon)}{\pi_\varepsilon}, \tag{26}$$

and using the fact that the no-arbitrage assumption implies,

$$\sum_{\varepsilon\in\{H,L\}} Q(\varepsilon) = Q,$$

we may write the lifecycle budget constraint of a worker,

$$\mathbb{E}\left[\tilde{p}(\varepsilon) c_i(\varepsilon) + \tilde{w}(\varepsilon)(1 - n_i(\varepsilon))\right] + rM_i \leq \mathcal{W}, \tag{27}$$

where workers' wealth under uncertainty is defined as

$$\mathcal{W} \equiv \mathbb{E}[\tilde{w}(\varepsilon)] + \left(\frac{B}{6} + \frac{rM}{3}\right). \tag{28}$$

Here $\mathbb{E}$ is the expectations operator, defined using the probability distribution $\{\pi_H, \pi_L\}$ and the term in the last bracket denotes the net transfer from the government.

10. Workers' Choices Under Uncertainty

Workers maximize the following logarithmic expected utility function,

$$\mathcal{U}_i = \mathbb{E}\left[\lambda_i \log(c_i(\varepsilon)) + \mu_i \log(1 - n_i(\varepsilon)) + \gamma_i \log\left(\frac{M}{w(\varepsilon)}\right)\right],$$

[9]Note that this definition implies $\tilde{w}(\varepsilon)/\tilde{p}(\varepsilon) = w(\varepsilon)/p(\varepsilon)$ which we shall exploit subsequently.

subject to the lifecycle budget constraint, Equation (27). The assumption that preferences are logarithmic implies, as in the certainty case, that budget shares are constant,

$$\tilde{w}(\varepsilon)(1-n_i(\varepsilon)) = \mu_i \mathcal{W}, \tag{29}$$

$$\tilde{p}(\varepsilon) c_i(\varepsilon) = \lambda_i \mathcal{W}, \tag{30}$$

$$rM_i = \gamma_i \mathcal{W}, \tag{31}$$

with wealth $\mathcal{W}$ defined in Equation (28). Rearranging terms in Equation (29) leads to the labor supply function for type i in state ε,

$$n_i(\varepsilon) = 1 - \mu_i \frac{\mathcal{W}}{\tilde{w}(\varepsilon)}. \tag{32}$$

If $w(H)$ is different from $w(L)$, the real value of the worker's net transfer will differ across states. One might think that this difference would be irrelevant in a complete markets environment because the relative security price, $Q(\varepsilon)/Q$, can adjust to reflect a change in the value of the unit of account. That argument breaks down in our model because entrepreneurs are unable to insure across alternative states of nature.

11. Entrepreneurs' Choices Under Uncertainty

Entrepreneurs solve two different problems, one for each realization of ε,

$$\max_{\{c_3(\varepsilon), n(\varepsilon)\}} U_3(\varepsilon) = \log(c_3(\varepsilon)),$$

subject to a budget constraint

$$c_3(\varepsilon) \leq \Pi(\varepsilon) - \frac{\mathcal{T}_3}{p(\varepsilon)}, \tag{33}$$

where real profits in state ε, $\Pi(\varepsilon)$, are defined as

$$\Pi(\varepsilon) \equiv (n(\varepsilon))^{\alpha} - \frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)} n(\varepsilon), \tag{34}$$

and

$$\mathcal{T}_3 \equiv \frac{(B - rM)}{3}, \tag{35}$$

is the money value of the entrepreneur's tax burden. The solution to these problems leads to the labor demand functions,

$$n(\varepsilon) = \left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{1}{\alpha-1}}, \tag{36}$$

the output supply functions,

$$y(\varepsilon) = \left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{\alpha}{\alpha-1}}, \tag{37}$$

and the consumption demand functions of entrepreneurs,

$$c_3(\varepsilon) = (1-\alpha)\left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{\alpha}{\alpha-1}} - \frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\frac{(B-rM)}{3w(\varepsilon)}. \tag{38}$$

Notice that $w(\varepsilon)$ enters Equation (38) independently of $\tilde{w}(\varepsilon)$. That follows from the assumption of incomplete participation in the asset markets.

12. Incomplete Participation Rational Expectations Equilibria

There are five goods: consumption in states H and L, leisure in states H and L and money. And there are four state-contingent prices, $\tilde{p}(H)$, $\tilde{p}(L)$, $\tilde{w}(H)$, $\tilde{w}(L)$, and one non state-contingent interest rate, r. Because entrepreneurs are not present in the asset market, their consumption demand functions, given by Equation (38), also depend on the dollar wages, $w(H)$ and $w(L)$. We will show that the appearance of $w(\varepsilon)$ in Equation (38) implies that non-fundamental uncertainty matters in our economy, even in the presence of complete insurance markets.

Using the definitions of λ and μ from Equation (19), we may write labor and goods market clearing equations for each state. The labor market clearing equation is given by the expression,

$$\left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{1}{\alpha-1}} = 2 - \mu\frac{\mathcal{W}}{\tilde{w}(\varepsilon)}, \tag{39}$$

with wealth $\mathcal{W}$ defined in Equation (28). To derive the goods market clearing condition we equate output supply from Equation (37) to the sum of workers consumption demand, from Equation (30), and entrepreneurs consumption demand, Equation (38). After rearranging terms and dividing by $\tilde{w}(\varepsilon)/\tilde{p}(\varepsilon)$ we arrive at Equation (40),

$$\left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{1}{\alpha-1}} = \lambda\frac{\mathcal{W}}{\tilde{w}(\varepsilon)} - \left(\frac{B - rM}{3w(\varepsilon)}\right). \tag{40}$$

The equality of the demand and supply of money, gives one additional equation,

$$M = \gamma\frac{\mathcal{W}}{r}. \tag{41}$$

To complete our characterization of a rational expectations equilibrium, we define the belief function,

$$w(\varepsilon) = \varphi(M,\varepsilon) \equiv M + \varepsilon. \tag{42}$$

We will use $w(\varepsilon)$ to characterize equilibrium beliefs in a rational expectations equilibrium.

Definition 2. *An incomplete participation rational expectations equilibrium are monetary and fiscal policies $\{M, B\}$, a belief function $\varphi(M,\varepsilon)$, an allocation $\{\{n_i(\varepsilon), M_i\}_{i\in 1,2}, \{c_i(\varepsilon)\}_{i=1,2,3}$, $y(\varepsilon), n(\varepsilon)\}_{\varepsilon\in\{H,L\}}$, a set of state-prices $\{\tilde{p}(\varepsilon), \tilde{w}(\varepsilon), r\}_{\varepsilon\in\{H,L\}}$, and a pair of Arrow security prices $\{Q(\varepsilon)\}_{\varepsilon\in\{H,L\}}$ that satisfies budget balance and optimality. An equilibrium price system is a non-negative 7-tuple $\{\tilde{p}(\varepsilon), \tilde{w}(\varepsilon), Q(\varepsilon), r\}_{\varepsilon\in\{H,L\}}$ such that equations (39) and (40) hold in each state ε, Equation (41) holds and the money wage in each state is given by the belief function, Equation (42).*

Proposition 2 characterizes the equilibrium corresponding to any given belief function. The belief function determines $w(L), w(H)$ and the market clearing equations determine $\tilde{w}(\varepsilon), \tilde{p}(\varepsilon)$, $Q(\varepsilon)$ and r. As in the perfect foresight case, the feasibility condition eliminates beliefs which translate into negative prices or allocations.

Proposition 2. *Let $\{M, B\} \geq 0$ characterize public sector policy, and let $\{w(L), w(H)\} > 0$ be wages implied by a belief function $\varphi(M,\varepsilon)$. Assume further that the feasibility condition holds* *in both states*

$$w(\varepsilon) \quad \geq \quad \frac{\mu_i B}{4(1+\lambda+\mu)-6\mu_i}, \quad i \in \{1,2\}, \quad \varepsilon \in \{L,H\} \tag{43}$$

$$w(\varepsilon) \quad \geq \quad \frac{2-\mu+\lambda(2-3\alpha)}{\mu+\alpha\lambda}\frac{B}{2}. \;\; \varepsilon \in \{L,H\} \tag{44}$$

Define the following constants $\theta, X_L, Y_L, X_H, Y_H, \theta_1$ *and* θ_2*,*

$$\theta \equiv \frac{\lambda+\mu}{\gamma}, \tag{45}$$

$$X_L \equiv [6w(L)+B], \quad X_H \equiv [6w(H)+B], \quad Y_L \equiv 3\pi_L\theta, \quad Y_H \equiv 3\pi_H\theta, \tag{46}$$

$$\theta_1 \equiv \frac{[X_H(1+Y_L)-X_L(1+Y_H)]}{X_L Y_H}, \quad \theta_2 \equiv \frac{X_H Y_L}{X_L Y_H}. \tag{47}$$

The equilibrium ratio of Arrow security prices $q \equiv Q(L)/Q(H)$*, is the unique positive solution to the quadratic equation*

$$q^2 - \theta_1 q - \theta_2 = 0. \tag{48}$$

The equilibrium Arrow security prices satisfy

$$Q(H) = \frac{(q+Y_L[q+1])M}{X_L q(1+q)+(1+q)(q+Y_L[q+1])M}, \tag{49}$$

and

$$Q(L) = qQ(H). \tag{50}$$

The equilibrium state wages $\tilde{w}(\varepsilon)$ *are given by the expressions*

$$\tilde{\omega}(L) = \frac{w(L)}{(1+q^{-1})\pi_L}, \quad \tilde{\omega}(H) = \frac{w(H)}{(1+q)\pi_H}, \tag{51}$$

and the state prices are equal to

$$\tilde{p}(\varepsilon) = \frac{\tilde{w}(\varepsilon)}{\alpha}\left(2-\frac{\mu\mathcal{W}}{\tilde{w}(\varepsilon)}\right)^{1-\alpha}, \quad \varepsilon \in \{L,H\}, \tag{52}$$

where

$$Q = Q(L)+Q(H), \quad r \equiv \frac{1-Q}{Q}, \quad \text{and} \quad \mathcal{W} = \frac{(1-Q)}{Q}M \quad . \tag{53}$$

Conditional on the $\tilde{w}(\varepsilon), \tilde{p}(\varepsilon)$ *and* $Q(\varepsilon)$ *characterized above, the equilibrium quantities* $c_i(\varepsilon), n_i(\varepsilon)$ *and* M_i *can then be determined from Equations (29) – (31), while* $n(\varepsilon), y(\varepsilon)$ *and* $c_3(\varepsilon)$ *are characterized in Equations (36) – (38).*

Proposition 2, proved in Appendix B, establishes a mapping from beliefs to equilibrium prices and allocations. The following corollary confirms that these beliefs don't only affect nominal prices but also the corresponding real allocations.

Corollary 3. *Whenever* $w(L) \neq w(H)$*,*

$$n_i(L) \neq n_i(H), \quad \text{and} \quad c_i(L) \neq c_i(H). \tag{54}$$

Corollary 3 is proved in Appendix C. This is an example, for a monetary economy, of Cass and Shell's (1983) result that, when there is incomplete asset market participation, "sunspots matter".

13. Pareto Optimality and Complete Participation

In this section we consider a counter-factual economy in which entrepreneurs are present in the asset markets that open before they are born. We show that, in this scenario, the equilibrium associated with *any* belief function has the property that

$$n_i(H) = n_i(L) \qquad \text{and} \qquad c_i(H) = c_i(L).$$

Thus, although there are still multiple equilibria in the complete participation case, sunspots cease to have real effects. One can additionally invoke the first welfare theorem to show that the resulting equilibrium is Pareto efficient, in contrast to the one characterized in the previous section.

We continue to assume that entrepreneurs only care about consumption and receive no part of the government transfer. If entrepreneurs are present in period 1, they will trade Arrow securities subject to the first period constraint,

$$\sum_{\varepsilon \in \{L,H\}} Q(\varepsilon) A_3(\varepsilon) = 0. \tag{55}$$

Maximizing profit in each state in period 2 leaves entrepreneurs with real pre-tax profits

$$\Pi(\varepsilon) = (1-\alpha)\left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{\alpha}{\alpha-1}}, \tag{56}$$

in each state and their consumption in period 2 is subject to the constraint

$$p(\varepsilon) c_3(\varepsilon) \leq p(\varepsilon)\Pi(\varepsilon) - \left(\frac{B - rM}{3}\right) + A_3(\varepsilon), \tag{57}$$

where

$$\left(\frac{B - rM}{3}\right) \equiv \mathcal{T}_3, \tag{58}$$

is the money value of the entrepreneur's tax liability. Substituting Inequality (57) into (55) and using the no arbitrage condition, $Q(L) + Q(H) = Q$, leads to the entrepreneur's lifecycle constraint,

$$\sum_{\varepsilon \in \{L,H\}} Q(\varepsilon) p(\varepsilon) c_3(\varepsilon) \leq \sum_{\varepsilon \in \{L,H\}} Q(\varepsilon) p(\varepsilon) \Pi(\varepsilon) - Q\left(\frac{B - rM}{3}\right). \tag{59}$$

Using the definition of $\tilde{p}(\varepsilon)$ we may rewrite Equation (59) as follows,

$$\mathbb{E}[\tilde{p}(\varepsilon) c_3(\varepsilon)] \leq \mathbb{E}[\tilde{p}(\varepsilon)\Pi(\varepsilon)] - \left(\frac{B - rM}{3}\right). \tag{60}$$

Expected utility maximization, subject to Inequality (60), leads to the solution

$$c_3(\varepsilon) = \frac{\mathbb{E}[\tilde{p}(\varepsilon)\Pi(\varepsilon)]}{\tilde{p}(\varepsilon)} - \frac{1}{\tilde{p}(\varepsilon)}\left(\frac{B - rM}{3}\right). \tag{61}$$

Notice, and this is important, that dollar prices $p(\varepsilon)$ or $w(\varepsilon)$ no longer separately appear in the entrepreneur's budget constraint, which can be expressed entirely using state prices. Although the entrepreneur makes the same labor demand and output supply decisions in each state, she does not make the same consumption decisions. Instead of consuming the after tax profit in each state, access to an insurance market allows the entrepreneur to smooth consumption across states.

14. Rational Expectations Equilibrium with Complete Participation

Using the labor demand and supply functions of each type, we may write the excess demand equations for the economy with complete asset market participation as follows. Recall first that $\mathcal{W}$, the wealth of a worker, is defined as

$$\mathcal{W} = \mathbb{E}[\tilde{w}(\varepsilon)] + \left(\frac{B}{6} + \frac{rM}{3}\right).$$

Using this definition, the labor market equilibrium condition is given by Equation (62),

$$\left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{1}{\alpha-1}} = 2 - \mu\frac{\mathcal{W}}{\tilde{w}(\varepsilon)}, \tag{62}$$

and the goods market equilibrium condition is,

$$\left(\frac{1}{\alpha}\frac{\tilde{w}(\varepsilon)}{\tilde{p}(\varepsilon)}\right)^{\frac{\alpha}{\alpha-1}} = \frac{1}{\tilde{p}(\varepsilon)}\left\{\mathbb{E}\left[\tilde{p}(\varepsilon)\Pi(\varepsilon)\right] - \left(\frac{B-rM}{3}\right)\right\} + \lambda\frac{\mathcal{W}}{\tilde{p}(\varepsilon)}. \tag{63}$$

Finally, equality of the demand and supply of money is represented by Equation (64).

$$\gamma\mathcal{W} = rM. \tag{64}$$

The important difference of the complete participation model from the incomplete participation model that we discussed in Section 12, is to be found in Equation (63), which no longer contains terms in $w(L)$ or $w(H)$. That fact implies that equations (62) and (63) are identical in states L and H.

Definition 3. *A complete participation rational expectations equilibrium is a monetary and a fiscal policy $\{M, B\}$, a belief function $\varphi(M,\varepsilon)$, an allocation $\{\{n_i(\varepsilon), M_i\}_{i\in 1,2}, \{c_i(\varepsilon)\}_{i=1,2,3}\ ,$ $y(\varepsilon), n(\varepsilon)\}_{\varepsilon\in\{H,L\}}$ and a set of state-dependent prices $\{\tilde{p}(\varepsilon), \tilde{w}(\varepsilon), Q(\varepsilon), r\}_{\varepsilon\in\{H,L\}}$ that satisfies budget balance and optimality. An equilibrium price system, $\{\tilde{p}(\varepsilon), \tilde{w}(\varepsilon), Q(\varepsilon), r\}_{\varepsilon\in\{H,L\}}$, is a non-negative 7-tuple such that equations (62) and (63) hold in each state, Equation (64) holds, and the money wage in each state is given by the belief function, Equation (42).*

Proposition 4. *Let $\{M, B\} \geq 0$ characterize public sector policy, and let $\{w(L), w(H)\} > 0$ be wages implied by a belief function $\varphi(M,\varepsilon)$ such that the feasibility constraints hold*

$$w(\varepsilon) \geq \frac{\mu_i B}{4(1+\lambda+\mu) - 6\mu_i}, \quad i\in\{1,2\}, \quad \varepsilon\in\{L,H\} \tag{65}$$

$$w(\varepsilon) \geq \frac{2-\mu+\lambda(2-3\alpha)}{\mu+\alpha\lambda}\frac{B}{2}. \quad \varepsilon\in\{L,H\} \tag{66}$$

Define $q \equiv Q(L)/Q(H)$. Then,

$$q = \frac{w(H)}{w(L)}\frac{\pi_L}{\pi_H}. \tag{67}$$

The equilibrium value of $Q(H)$ is given by the expression

$$Q_H = \frac{M(6-2\gamma)\pi_H}{(1+q)(M(6-2\gamma)+B\gamma)\pi_H + 6\gamma w(H)} \tag{68}$$

with

$$Q(L) = qQ(H) \qquad \textit{and} \qquad Q = Q(L) + Q(H).$$

The equilibrium state wages $\tilde{w}(\varepsilon)$ are given by the expressions

$$\tilde{w}(L) = \frac{w(L)}{(1+q^{-1})\,\pi_L}, \qquad \tilde{w}(H) = \frac{w(H)}{(1+q)\,\pi_H}, \tag{69}$$

and the equilibrium state price of consumption goods $\tilde{p}(\varepsilon)$ are equal to

$$\tilde{p}(\varepsilon) = \frac{\tilde{w}(\varepsilon)}{\alpha}\left(2 - \frac{\mu\mathcal{W}}{\tilde{w}(\varepsilon)}\right)^{1-\alpha}. \tag{70}$$

Conditional on the $\tilde{w}(\varepsilon), \tilde{p}(\varepsilon)$ and $Q(\varepsilon)$ characterized above, the equilibrium quantities $c_i(\varepsilon), n_i(\varepsilon)$ and M_i can then be solved for from Equations (39) – (41), while $n(\varepsilon), y(\varepsilon)$ and $c_3(\varepsilon)$ are characterized in Equations (36) – (37) and 61, respectively. □

See Appendix D for a proof of this proposition.

Corollary 5. *Under full participation, the equilibrium associated with* any *belief function has the property that*

$$n_i(L) = n_i(H), \quad i \in \{1,2\} \qquad \textit{and} \qquad c_i(L) = c_i(H), \quad i \in \{1,2,3\}. \tag{71}$$

Proof. The proof follows directly from the proposition above: the formula for $Q(\varepsilon)$ implies that state-wages $\tilde{w}(\varepsilon)$ are the same in both states, and thus, by definition, so are state-prices $\tilde{p}(\varepsilon)$. The solutions to the workers' optimization problems, equations (39) – (41), then show that the corresponding real allocations are state-invariant. This establishes that there is complete consumption and leisure insurance. ∎

To clarify what is happening in the model, we now characterize the entrepreneur's asset portfolio.

Proposition 6. *In the full participation model, agent 3's asset position is given by*

$$A_3(H) = (1-\pi_H)\left(\frac{B - rM}{3}\right)\left(1 - \frac{w(H)}{w(L)}\right), \quad A_3(L) = \pi_H\left(\frac{B - rM}{3}\right)\left(1 - \frac{w(L)}{w(H)}\right). \tag{72}$$

Proposition 6 is proved in Appendix E.

An immediate implication of this proposition and the fact that $w(H) > w(L)$ is that $A_3(H)$ is negative, while $A_3(L)$ is positive. Agent 3 uses the asset market to buy insurance from the workers against the $w(L)$ outcome and to sell insurance to the workers against the $w(H)$ outcome.

When agent 3 is excluded from trade in Arrow securities, she is positively affected by higher $w(\varepsilon)$ for two reasons. First, agent 3 pays for part of the nominal transfer which workers receive from the government. Higher nominal wages mean that the *real* value of the transfer is lower which makes agent 3 better off when $\varepsilon = H$. Second, the fact that workers are poorer in state $\varepsilon = H$ means that they consume less leisure. Equilibrium employment is higher and so are output and the real value of profits that accrue to entrepreneurs. In contrast, workers are worse off in state $\varepsilon = H$ and hence both groups of agents will trade Arrow securities up to the point at which their *real* consumption and leisure are constant across states.

15. Nominal Bond and Equity Portfolios

In this section we translate the abstract notion of Arrow securities into the more familiar case in which agents cross-insure using debt and equity. We assume that a nominal bond pays a dollar in every state ε, while equities entitle their owners to a share of the entrepreneur's nominal profit stream, which we denote with the symbol $\tilde{\Pi}(\varepsilon)$ to distinguish it from the real profit stream, $\Pi(\varepsilon)$,

$$\tilde{\Pi}(\varepsilon) \equiv p(\varepsilon)\, y(\varepsilon) - w(\varepsilon) n(\varepsilon).$$

Using these definitions we prove the following proposition.

Proposition 7. *In the full participation model, agent 3 purchases nominal bonds with a face value of*

$$B_3 \equiv \frac{B - rM}{3}, \tag{73}$$

where

$$r = \frac{1-Q}{Q}. \tag{74}$$

The purchase of bonds by entrepreneurs is financed by selling a share λ of the entrepreneur's profit stream where

$$\lambda = \frac{QB}{Q(L)\,\tilde{\Pi}(L) + Q(H)\,\tilde{\Pi}(H)}. \tag{75}$$

Proof. See Appendix F. ∎

This proposition establishes that the entrepreneur uses nominal bonds to insure herself against volatility in real tax expenditures. In equilibrium, fluctuations in the nominal price level cause fluctuations in the real value of tax liabilities that are perfectly insured by the nominal bonds she purchases from workers. Workers provide this insurance by purchasing a share in the firm. This share is risk free because fluctuations in the nominal profit stream are offset, in equilibrium, by fluctuations in the price level.

The equilibrium with complete participation Pareto dominates the equilibrium in the absence of complete participation because it provides an additional opportunity for trade. In Section 16 we show how the government can restore Pareto efficiency, even if entrepreneurs are not present in period 1, by trading on their behalf.

16. Implementing a Pareto Optimal Equilibrium

In this Section, we return to the case in which entrepreneurs are excluded from participating in asset trades, and we assume that the treasury makes dollar denominated lump-sum transfers worth $QB/2$ to types 1 and 2 paid for by issuing nominal debt with a face value of B. In this environment, there are redundant assets since debt and equity can be replicated by trading in Arrow securities.

A bond is a claim to B dollars in state ε and it can be replicated by the purchase of B Arrow securities of type L and B securities of type H. Equity, is a claim to $\lambda\tilde{\Pi}(\varepsilon)$ dollars in state ε that can be replicated by the purchase of a portfolio of $\lambda\tilde{\Pi}(L)^{-1}$ securities of type L and $\lambda\tilde{\Pi}(H)^{-1}$ securities of type H where λ is a share of the firm.

We assume that private agents purchase Arrow securities and they do not trade in riskless bonds and we continue to assume that the central bank, purchases debt A_{CB} where,

$$M = QA_{CB}.$$

In addition, we allow the bank to make supplementary trades in debt and equity, subject to the constraint that these supplementary security purchases are self-financing,

$$M = QA_{CB} + Q\tilde{A}_{CB} + P_E\lambda_{CB}, \tag{76}$$

where the self-financing constraint implies that

$$Q\tilde{A}_{CB} + P_E\lambda_{CB} = 0. \tag{77}$$

Here, $\tilde{A}_{CB}$ are additional purchases of debt by the central bank that may be positive or negative, λ_{CB} is the number of shares to the nominal profit stream that is bought or sold by the central bank and

$$P_E = Q(L)\tilde{\Pi}(L) + Q(H)\tilde{\Pi}(H), \tag{78}$$

is the price of a claim to a share in the firm. We allow short sales so that λ_{CB} may be negative.

Let $\mathcal{S}$ denote seigniorage revenues associated with money issuance and define

$$\tilde{\mathcal{S}}(\varepsilon) = \mathcal{S} + \left[\tilde{\Pi}(\varepsilon) - \tilde{A}_{CB}\right], \tag{79}$$

where

$$\left[\tilde{\Pi}(\varepsilon) - \tilde{A}_{CB}\right], \tag{80}$$

is the additional profit or loss associated with the risky component of the bank's balance sheet. We then arrive at the balance sheet of the central bank presented in Table 2.

Assets	Liabilities
QA_{CB}	M
$Q\mathcal{S}$	
$Q\tilde{A}_{CB}$	$P_E\lambda_{CB}$
	E_{CB}

Table 2. The Central Bank Balance Sheet

As in our previous model, the seigniorage revenues from money creation are repaid to the treasury. However, there is now risk associated with the central bank's balance sheet. The modified definition of an equilibrium accounts for the fact that the central bank trades in the asset markets. And in Proposition 8, we establish that there exists a set of central bank trades that leads to the same real allocations as those in the complete participation equilibrium described in Proposition 4.

Definition 4. *A rational expectations equilibrium with a self-financing stabilization policy is a monetary and a fiscal policy $\{M, B, A_{CB}, \lambda_{CB}\}$, a belief function $\varphi(M,\varepsilon)$, an allocation $\{\{n_i(\varepsilon), M_i\}_{i\in 1,2}, \{c_i(\varepsilon)\}_{i=1,2,3}\ ,\ y(\varepsilon), n(\varepsilon)\}_{\varepsilon\in\{H,L\}}$ and a set of state-dependent prices $\{\tilde{p}(\varepsilon), \tilde{w}(\varepsilon), Q(\varepsilon), r\}_{\varepsilon\in\{H,L\}}$ that satisfies budget balance and optimality and the self-financing condition*

$$Q\tilde{A}_{CB} + \lambda_{CB}P_E = 0, \tag{81}$$

where

$$P_E = Q(L)\tilde{\Pi}(L) + Q(H)\tilde{\Pi}(H). \tag{82}$$

An equilibrium price system, $\{\tilde{p}(\varepsilon), \tilde{w}(\varepsilon), Q(\varepsilon), r\}_{\varepsilon \in \{H,L\}}$, is a non-negative 7-tuple such that equations (62) and (63) hold in each state, Equation (64) holds, and the money wage in each state is given by the belief function, Equation (42).

Proposition 8. *Let $\left\{M, B, \tilde{A}_{CB}, \lambda_{CB}\right\} \geq 0$ characterize public sector policy, and let $\{w(L), w(H)\} > 0$ be wages implied by a belief function $\varphi(M, \varepsilon)$ such that the feasibility constraints hold*

$$w(\varepsilon) \geq \frac{\mu_i B}{4(1+\lambda+\mu) - 6\mu_i}, \quad i \in \{1,2\}, \quad \varepsilon \in \{L,H\} \tag{83}$$

$$w(\varepsilon) \geq \frac{2-\mu+\lambda(2-3\alpha)}{\mu+\alpha\lambda}\frac{B}{2}. \quad \varepsilon \in \{L,H\} \tag{84}$$

Let $\left\{\tilde{A}_{CB}, \lambda_{CB}\right\}$ be a self-financing policy where,

$$\frac{\mathcal{T}(L)}{w(L)} = \frac{\mathcal{T}(H)}{w(H)}. \tag{85}$$

There exists a unique equilibrium in which allocations are the same as those implemented in the complete participation rational expectations equilibrium. The policy is implemented by a policy in which the central bank buys debt equal to $\tilde{A}_{CB}$ financed by selling equities λ_{CB} where

$$\tilde{A}_{CB} = B - rM, \quad \text{and} \quad \lambda_{CB} = -Q\left(\frac{B - rB}{Q(L)\tilde{\Pi}(L) + Q(H)\tilde{\Pi}(H)}\right). \tag{86}$$

The prices $Q(L)$, $Q(H)$ and $r = (1-Q)/Q$ and the money value of profits in each state $\tilde{\Pi}(L)$ and $\tilde{\Pi}(H)$, are the values defined in Proposition 6.

Proof. See Appendix G. ∎

The fact that the equilibrium allocations are identical to those under complete participation means that the central bank is able to restore efficiency. In the proof of the proposition we establish that the central bank's position in the asset markets is three times the position that would be taken by the entrepreneur in the counter-factual complete markets equilibrium. Hence the workers' portfolios of risky assets will be larger than under complete participation. In both cases, the real value of the worker's and entrepreneurs after tax incomes are stabilized in the Pareto efficient equilibrium.

Corollary 9. *In the stabilization equilibrium, the returns on a real indexed bond is equal to the real return from holding equity.*

Proof. In the Pareto efficient equilibrium the real value of profit is the same in both states. It follows immediately that the return to an indexed bond is the same as the return to an equity share . ∎

This corollary implies that, when all uncertainty is non-fundamental, the central bank can implement the optimal policy by standing ready to trade indexed bonds at the same price as claims to the stock market. By doing so, it would end up holding the same asset portfolio $\left\{\tilde{A}_{CB}, \lambda_{CB}\right\}$ as the one described in Proposition 8.

17. Summary

Willem Buiter made the distinction between Quantitative Easing, defined as an increase in the size of the central bank balance sheet, and Qualitative Easing, defined as a change in its risk composition. In this paper we have provided a theory that explains the channel by which Qualitative Easing influences asset prices. According to our explanation, some asset price fluctuations are Pareto inefficient because people cannot insure themselves against the state of the world they are born into. By trading debt for equity in the asset markets, the central bank can make trades that stabilize Pareto inefficient asset price movements and that can potentially make everyone better off.

Our explanation builds on the idea that general equilibrium models of money always contain multiple equilibria. We believe that this idea is important not just in theory, but also in practice. In our view, a significant portion of asset price fluctuations in the real world are caused by self-fulfilling shifts from one equilibrium to another that cause inefficient shifts of wealth. Although we have explained our case in a simple two-period model, multiple equilibria are endemic to monetary models and our argument is much more general than the model that we have used to explain it. The fact that asset price volatility is Pareto inefficient provides, we believe, a strong case to make Qualitative easing a permanent component of future financial policy.

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Appendix A. Proof of Proposition 1

Proof. Combining the definition of wealth (2) with the money market clearing condition, Equation (22), we have the following definition of wealth that must hold in equilibrium

$$\mathcal{W} = \frac{6w + B}{2\,(1 + \lambda + \mu)}. \tag{A1}$$

Feasibility requires that

$$n_i = 2 - \mu_i \frac{\mathcal{W}}{w} > 0, \tag{A2}$$

for both types, which implies the first feasibility condition, Equation (A3).

$$w > \frac{\mu_i B}{4\,(1 + \lambda + \mu) - 6\mu_i}, \qquad i \in \{1, 2\} \tag{A3}$$

To derive the expression for the equilibrium value of r, we use the money market equilibrium condition, Equation (22), together with the equilibrium value of wealth from (A1). The real wage follows directly from inverting the labor market clearing equation.

For a valid equilibrium we also require that

$$c_3 = (1-\alpha)\left(\frac{1}{\alpha}\frac{w}{p}\right)^{\frac{\alpha}{\alpha-1}} - \frac{(B-rM)}{3p} > 0. \tag{A4}$$

Using equilibrium prices and wealth from equations (24), and the market clearing equations, (16) – (18), evaluated at equilibrium prices, we arrive at the second feasibility condition,

$$w \geq \frac{2-\mu+\lambda(2-3\alpha)}{\mu+\alpha\lambda}\frac{B}{2}. \tag{A5}$$

∎

Appendix B. Proof of Proposition 2

Proof. We begin with three facts that follow from the definitions of market clearing, Equations (39) – (41). First, from money market clearing,

$$\mathcal{W} = \frac{(1-Q)\,M}{Q}\frac{1}{\gamma} \equiv \frac{x}{\gamma}, \tag{B1}$$

where we define

$$x \equiv \frac{(1-Q)\,M}{Q} \equiv rM. \tag{B2}$$

Second, putting together the labor market and goods market clearing equations for each state, equations (39) and (40), and using the definition of x from Equation (B2) and the money market clearing equation, (B1), we have that,

$$2 - \frac{\theta x}{\tilde{w}(\varepsilon)} = \frac{x}{3w(\varepsilon)} - \frac{B}{3w(\varepsilon)}, \quad \varepsilon \in \{L, H\}, \tag{B3}$$

where we define

$$\theta \equiv \frac{\lambda+\mu}{\gamma}. \tag{B4}$$

Third, we use the definition of $\tilde{w}(\varepsilon)$,

$$\tilde{w}(\varepsilon) \equiv \frac{Q(\varepsilon)\,w(\varepsilon)}{Q\pi_\varepsilon}, \quad \varepsilon \in \{L, H\}. \tag{B5}$$

Substituting (B5) into (B3) gives,

$$2 - \frac{Q\pi_\varepsilon \theta x}{Q(\varepsilon)\,w(\varepsilon)} = \frac{x}{3w(\varepsilon)} - \frac{B}{3w(\varepsilon)}, \quad \varepsilon \in \{L, H\}. \tag{B6}$$

Rearranging this equation and using the no arbitrage condition, $Q = Q(L) + Q(H)$ leads to the following expression for x

$$x = \frac{[6w(L)+B]\,Q(L)}{Q(L)+3\pi_L\theta\,[Q(L)+Q(H)]} = \frac{[6w(H)+B]\,Q(H)}{Q(H)+3\pi_H\theta\,[Q(L)+Q(H)]}. \tag{B7}$$

Next define $q \equiv Q(L)/Q(H)$, and divide the numerator and denominator of Equation (B7) by $Q(H)$ to give

$$\frac{[6w(L)+B]\,q}{q+3\pi_L\theta\,[1+q]} = \frac{[6w(H)+B]}{1+3\pi_H\theta\,[1+q]}. \tag{B8}$$

Rearranging this equation and defining

$$X_L \equiv [6w(L)+B], \quad X_H \equiv [6w(H)+B], \quad Y_L = 3\pi_L\theta, \quad Y_H \equiv 3\pi_H\theta, \tag{B9}$$

$$X_L q\,[1+Y_H(1+q)] = X_H\,[q+Y_L(1+q)], \tag{B10}$$

and

$$\theta_1 \equiv \frac{[X_H (1 + Y_L) - X_L (1 + Y_H)]}{X_L Y_H}, \quad \theta_2 \equiv \frac{X_H Y_L}{X_L Y_H}, \tag{B11}$$

gives the following quadratic equation in q,

$$q^2 - \theta_1 q - \theta_2 = 0. \tag{B12}$$

This is Equation (48) in Proposition 2. Let r_1 and r_2 be the roots of this quadratic and note that r_1 and r_2 are given by the expressions

$$r_i = \frac{1}{2}\left(\theta_1 \pm \sqrt{\theta_1^2 + 4\theta_2}\right). \tag{B13}$$

It follows that both roots are real and that there is a unique non-negative root.

Next note that

$$x \equiv M\frac{(1-Q)}{Q} \equiv \left(\frac{1}{Q_H} - 1 - q\right)\frac{M}{1+q}, \tag{B14}$$

and use Equation (B8) to write

$$x = \frac{X_L q}{q + Y_L (1+q)}. \tag{B15}$$

Combining equations (B15) and (B14) leads to the expression for $Q(H)$, Equation (49) in Proposition 2. Equation (50) follows from the definition of Q. To derive Equations (51) use the definitions of $w(\varepsilon)$, Equations (26).

Equation (52) follows from the labor market clearing equation and (53) from the (B1) and the no arbitrage assumption. It remains to check that Inequality (43) is sufficient to guarantee that labor supply is feasible and that (44) guarantees that entrepreneurs' consumption is non-negative. That follows from the fact that theses assumptions guarantee feasibility in each state individually and therefore feasibility in a convex combination of the states as well. ∎

Appendix C. Proof of Corollary 3

Proof. Labor supply of person i, for $i \in \{1, 2\}$ is given by the expression

$$n_i(\varepsilon) = 1 - \mu_i \frac{\mathcal{W}}{\tilde{w}(\varepsilon)}, \tag{C1}$$

and consumption by

$$c_i(\varepsilon) = \lambda_i \frac{\mathcal{W}}{\tilde{w}(\varepsilon)}. \tag{C2}$$

Hence to establish inequalities (54) we need only show that

$$\tilde{w}(L) \neq \tilde{w}(H). \tag{C3}$$

But from Equation (B3) we have that

$$\frac{\theta x}{\tilde{w}(\varepsilon)} = \frac{B}{3w(\varepsilon)} - \frac{x}{3w(\varepsilon)} - 2, \tag{C4}$$

from which it follows that $\tilde{w}(L) = \tilde{w}(H)$ if and only if, $w(L) \neq w(H)$. The inequality of the consumption of entrepreneurs across states follows from the fact that their income is a function of the real wage. ∎

Appendix D. Proof of Proposition 4

Proof. Combining labor market equilibrium, Equation (62) with goods market equilibrium (63) leads to

$$\frac{1}{\alpha}\left(2-\mu\frac{\mathcal{W}}{\tilde{w}\left(\varepsilon\right)}\right)=\frac{1}{\tilde{w}\left(\varepsilon\right)}\left\{\mathbb{E}\left[\tilde{p}\left(\varepsilon\right)\Pi\left(\varepsilon\right)\right]-\left(\frac{B-rM}{3}\right)\right\}+\lambda\frac{\mathcal{W}}{\tilde{w}\left(\varepsilon\right)},\qquad\varepsilon\in\left\{L,H\right\}.\tag{D1}$$

Because these two state equations are identical (the term in the wiggly brackets is a constant) it immediately follows that

$$\tilde{\omega}\left(L\right)=\tilde{w}\left(H\right)\equiv\tilde{w}.\tag{D2}$$

Using this fact, and the definition of $\tilde{w}\left(\varepsilon\right)$ gives,

$$q\equiv\frac{Q\left(L\right)}{Q\left(H\right)}=\frac{w\left(H\right)\pi_L}{w\left(L\right)\pi_H}.\tag{D3}$$

This establishes the expression for q, Equation (67), in the statement of Proposition 4.

Next we seek expressions for $Q\left(L\right)$ and $Q\left(H\right)$ individually. Combining the definition of workers wealth, Equation (28) with the money market equilibrium condition, Equation (64), and using Equation (D2) gives the following equation linking $\tilde{\omega}$ and r,

$$\mathcal{W}=\frac{rM}{\gamma}=\tilde{w}+\frac{B}{6}+\frac{rM}{3}\tag{D4}$$

Note also that

$$\tilde{w}=\frac{w\left(H\right)Q\left(H\right)}{Q\pi_H}.\tag{D5}$$

The no arbitrage condition, $Q=Q\left(L\right)+Q\left(H\right)$ implies that

$$\frac{Q\left(H\right)}{Q}=\frac{1}{1+q}.\tag{D6}$$

Using no arbitrage and the definition of r we also have that

$$r=\frac{1-Q}{Q}=\frac{1-Q_L-Q_H}{Q_L+Q_H}=\frac{\frac{1}{Q_H}-\left(1+q\right)}{1+q},\tag{D7}$$

which simplifies to give,

$$Q_H=\frac{1}{\left(1+r\right)\left(1+q\right)}.\tag{D8}$$

Next, we rearrange Equation (D4)

$$rM\left(\frac{3-\gamma}{\gamma}\right)=3\frac{w\left(H\right)}{\left(1+q\right)\pi_H}+\frac{B}{2},\tag{D9}$$

to find the following expression for $(1+r)$

$$\left(1+r\right)=\left(M\left(\frac{3-\gamma}{\gamma}\right)+3\frac{w\left(H\right)}{\left(1+q\right)\pi_H}+\frac{B}{2}\right)\left(M\left(\frac{3-\gamma}{\gamma}\right)\right)^{-1}.\tag{D10}$$

Finally, combining (D10) with (D8) and using the definition of γ gives Equation (68) in the statement of Proposition 4, which is the expression we seek

$$Q_H=\frac{M\left(6-2\gamma\right)\pi_H}{\left(1+q\right)\left(M\left(6-2\gamma\right)+B\gamma\right)\pi_H+6\gamma w\left(H\right)}.\tag{D11}$$

Equations (69) follow immediately from the definitions of state wages and Equation (70) follows from labor market clearing. The feasibility conditions, Inequalities (65) and (66) guarantee that labor supply for each worker and the consumption of entrepreneurs are each non-negative in equilibrium. ∎

Appendix E. Proof of Proposition 6

Proof. From the entrepreneur's budget constraint, Equation (57),

$$p(\varepsilon)\, c_3(\varepsilon) \leq p(\varepsilon)\, \Pi(\varepsilon) - \left(\frac{B - rM}{3}\right) + A_3(\varepsilon). \tag{E1}$$

From the solution to the entrepreneur's problem, we have that

$$c_3(\varepsilon) = \frac{\mathbb{E}\left[\tilde{p}(\varepsilon)\, \Pi(\varepsilon)\right]}{\tilde{p}(\varepsilon)} - \frac{1}{\tilde{p}(\varepsilon)} \left(\frac{B - rM}{3}\right). \tag{E2}$$

But from Proposition 4, $\tilde{p}(\varepsilon)$ is the same in both states and thus,

$$c_3(\varepsilon) = \Pi(\varepsilon) - \frac{1}{\tilde{p}(\varepsilon)} \left(\frac{B - rM}{3}\right). \tag{E3}$$

Rearranging Equation (E1) and combining it with (E3) gives the following expression,

$$A_3(\varepsilon) = \left(\frac{B - rM}{3}\right)\left(1 - \frac{p(\varepsilon)}{\tilde{p}(\varepsilon)}\right). \tag{E4}$$

Finally, from the definitions of $\tilde{p}(\varepsilon)$ and q we have that

$$\frac{p(H)}{\tilde{p}(H)} = \frac{\pi_H}{1+q}, \qquad \frac{p(L)}{\tilde{p}(L)} = \frac{q\pi_L}{1+q}. \tag{E5}$$

Combining equations (E4) and (E5)and using the facts that

$$q = \frac{w(H)}{w(L)} \frac{\pi_L}{\pi_H} = \frac{w(H)}{w(L)} \frac{1 - \pi_H}{\pi_h}, \tag{E6}$$

gives

$$A_3(H) = (1 - \pi_H)\left(\frac{B - rM}{3}\right)\left(1 - \frac{w(H)}{w(L)}\right), \quad A_3(L) = \pi_H \left(\frac{B - rM}{3}\right)\left(1 - \frac{w(L)}{w(H)}\right), \tag{E7}$$

which is the expression we seek. ∎

Appendix F. Proof of Proposition 7

Proof. By purchasing bonds with face value

$$\frac{B - rM}{3}, \tag{F1}$$

the entrepreneur consumes

$$c_3(\varepsilon) = \frac{\tilde{\Pi}(\varepsilon)}{p(\varepsilon)} - \frac{(B - rM)}{3p(\varepsilon)} + \left[\frac{B_3}{p(\varepsilon)} - \lambda \frac{\tilde{\Pi}(\varepsilon)}{p(\varepsilon)}\right], \tag{F2}$$

where

$$A_3(\varepsilon) \equiv B_3 - \lambda \tilde{\Pi}(\varepsilon), \tag{F3}$$

is the dollar value of Arrow securities in state ε. This is equal to the face value of debt, B_3, minus the share of profits, $\lambda \Pi(\varepsilon)$ that was sold to finance the purchase of debt. We established in Proposition 4, that $\tilde{\Pi}(\varepsilon)/p(\varepsilon)$ is the same in both states. It follows that if

$$B_3 = \frac{B - rM}{3}, \tag{F4}$$

that the entrepreneurs consumption is independent of the state. The share of profits that the entrepreneur sells to workers, λ, is defined by the entrepreneur's budget constraint in period 1,

$$QB_3 - \lambda P_E = 0, \tag{F5}$$

where

$$P_E = Q(L)\tilde{\Pi}(L) + Q(H)\tilde{\Pi}(H), \tag{F6}$$

is the price of a claim to the money value of profits. ∎

Appendix G. Proof of Proposition 8

To prove this proposition we show first that, if the security prices $Q(L)$ and $Q(H)$ and the nominal profit streams $\tilde{\Pi}(L)$ and $\tilde{\Pi}(H)$ are equal to the equilibrium values defined in Proposition 6 then the portfolio defined by Equation (86) stabilizes the real value of tax revenues.

In state ε the tax revenue levied by the treasury is given by the expression,

$$\mathcal{T}(\varepsilon) = [B - \mathcal{S}] - \left[\tilde{A}_{CB} + \lambda_{CB}\tilde{\Pi}(\varepsilon)\right]. \tag{G1}$$

To stabilize the real value of tax revenues the central bank must take a net asset position such that

$$\frac{[B - \mathcal{S}] - \left[\tilde{A}_{CB} + \lambda_{CB}\tilde{\Pi}(L)\right]}{w(L)} = \frac{[B - \mathcal{S}] - \left[\tilde{A}_{CB} + \lambda_{CB}\tilde{\Pi}(H)\right]}{w(H)}. \tag{G2}$$

By holding additional bonds equal to

$$\tilde{A}_{CB} = [B - S], \tag{G3}$$

Equation (G2) gives

$$\mathcal{T}(L) \equiv \lambda_{CB}\frac{\tilde{\Pi}(L)}{w(L)} = \lambda_{CB}\frac{\tilde{\Pi}(H)}{w(H)} \equiv \mathcal{T}(H). \tag{G4}$$

But from by definition,

$$\tilde{\Pi}(\varepsilon) = \Pi(\varepsilon)p(\varepsilon) = \Pi(\varepsilon)w(\varepsilon)\frac{\tilde{p}(\varepsilon)}{\tilde{w}(\varepsilon)}, \tag{G5}$$

where the last equality follows since

$$\frac{p(\varepsilon)}{w(\varepsilon)} = \frac{\tilde{p}(\varepsilon)}{\tilde{w}(\varepsilon)}. \tag{G6}$$

Combining these expressions gives

$$\mathcal{T}(L) \equiv \lambda_{CB}\Pi(L)\frac{\tilde{p}(L)}{\tilde{w}(L)} = \lambda_{CB}\Pi(H)\frac{\tilde{p}(H)}{\tilde{w}(H)} \equiv \mathcal{T}(H). \tag{G7}$$

But from Proposition 6, $\Pi(H) = \Pi(L)$. Hence the portfolio

$$\tilde{A}_{CB} = B - rM, \quad \text{and} \qquad \lambda_{CB} = -Q\left(\frac{B - rB}{Q(L)\tilde{\Pi}(L) + Q(H)\tilde{\Pi}(H)}\right), \tag{G8}$$

is self-financing and stabilizes the real value if tax revenues as claimed.

Next we establish that this tax policy generates the same after tax wealth positions for entrepreneurs and workers that they would choose if entrepreneurs could self insure. We showed in Proposition that entrepreneurs would choose to hold debt equal to

$$B_3 = \frac{B - rM}{3}. \tag{G9}$$

In the counter-factual complete participation equilibrium the wealth of the entrepreneur is equal to

$$\Pi(\varepsilon) - \frac{1}{3}\left(\frac{B - rM}{w(\varepsilon)}\right) + \left(\frac{B_3 - \lambda P_E}{w(\varepsilon)}\right), \tag{G10}$$

where

$$\frac{1}{3}\left(\frac{B - rM}{w(\varepsilon)}\right), \tag{G11}$$

is the real value of her tax obligation and

$$\{B_3, \lambda P_E\}, \tag{G12}$$

is the debt and equity portfolio that she takes to offset fluctuations in after-tax wealth.

In contrast, in the equilibrium with policy stabilization, the after tax wealth of the entrepreneur is

$$\Pi(\varepsilon) - \frac{1}{3}\left(\frac{B - rM}{w(\varepsilon)}\right) + \frac{1}{3}\left(\frac{\tilde{A}_{CB} - \lambda_{CB} P_E}{w(\varepsilon)}\right). \tag{G13}$$

It follows immediately that if the central bank chooses a policy where

$$\tilde{A}_{CB} = 3B_3 = B - rM, \tag{G14}$$

then the after tax wealth of the entrepreneur is identical in the equilibrium with stabilization as in the counter-factual complete markets equilibrium. It follows from Walras law that stabilizing the entrepreneurs income at its complete participation value also stabilizes the workers wealth at its complete participation value.