Secular Stagnation, Rational Bubbles, and Fiscal Policy

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Abstract

It is well known that rational bubbles can be sustained in balanced growth path of a deterministic economy when the return to capital r is equal to the growth rate g. When there is a lack of stores of value, bubbles can implement an efficient allocation. This paper considers a world where r fluctuates over time due to shocks to the marginal productivity of capital. Then, bubbles further efficiency, though they cannot implement first best. While bubbles can only be sustained when $r = g$ in a deterministic economy, $r > g$ "on average" in a stochastic economy. Fiscal policy improves welfare by adding an extra asset. Where only the elderly contribute to shifting resources between investment and consumption in a bubbly economy, fiscal policy allows part of that burden to be shifted to the young. Contrary to common wisdom, trade in bubbly assets implements *inter*generational transfers, while fiscal policy implements *intra*generational transfers. Hence, while bubbles and fiscal policy are perfect substitutes in the deterministic economy, fiscal policy dominates bubbles in a stochastic economy. For plausible parameter values, a higher degree of dynamic inefficiency should lead to a higher sovereign debt.

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1 Introduction

"There is increasing concern that we may be in an era of secular stagnation in which there is insufficient investment demand to absorb all the financial savings done by households and corporations, even with interest rates so low as to risk financial bubbles."

Lawrence Summers, Boston Globe, April 11, 2014

Real interest rates have come down steadily over the past thirty years. This phenomenon has been dubbed *secular stagnation*, see Richard Baldwin and Coen Teulings (2014) and IMF (2014, Chapter 3) for an overview of this debate. High precautionary saving in China, lower fertility and the increase in life expectancy have increased the supply of savings. Lower growth, the steady drop in the prices of capital goods, and a shift of economic activity towards IT with a low demand for capital have reduced investment demand. These factors have caused a worldwide decline in real interest rates, which has led to bubbles in asset prices. This paper addresses the question whether bubbles add to the resilience of the economy. We analyze this problem in a Walrasian world, where all markets are perfectly competitive, where Say's Law always holds and where expectations are rational. We assume that monetary authorities, by some divine touch, are able to avoid the zero lower bound for the nominal interest rate. The only missing market is that for intergenerational transfers. Bubbles are shown to partly fill the gap of this missing market.

Jean Tirole's (1985) celebrated paper on the feasibility of rational bubbles is the starting point of our analysis. A bubbly asset is defined as an asset that commands a higher price than the NPV of its expected future dividends. Tirole considers an overlapping generations model similar to that of Peter Diamond (1965) and Olivier Blanchard (1985). He shows that rational bubbles can be sustained along a balance growth path when the return on capital r is equal to the real growth rate g. Suppose youngsters save a fixed share of their income for future consumption. They can either invest in capital for a return r or buy a bubbly asset to be sold to the next generation. Since saving grows at rate g along a balanced growth path, spending on the bubbly asset grows at a rate g. When the supply of bubbly assets is fixed, its price therefore increases at a rate g. If $r > g$, buying bubbly assets would not be a profitable strategy. If $r < g$, nobody would invest in capital and its return would go up till either $r = g$ or there is no investment in capital at all. The condition $r < g$ is Peter Diamond's (1965) condition for dynamic inefficiency; $r < g$ is also Henry Aaron's (1966) condition for Pay-As-You-Go (PAYG) pension systems to be more efficient than funded systems. Bubbles are a substitute for PAYG pensions: the young pay the old either by buying their bubbly asset, or by the government taxing them to pay pensions to the old. Though widely different from a financial point of view, both institutions yield the same allocation of resources.

The contribution of this paper is to consider the role of bubbles when r varies over time due to shocks to the marginal productivity of capital, such that the economy jumps back and forth between $r = g$ and $r > g$. We analyze the potential role of fiscal policy in this type of world. In Tirole's balanced growth analysis analysis r is constant over time. Hence, either buying bubbly assets is attractive ($r = g$), or it is not ($r > g$). The economy does not have to jump back and forth between both states along a balanced growth path. In an economy with shocks to the return to capital, jumping back and forth between both regimes might be desirable. However, it poses its own problems. The problem is similar to switching back and forth from a funded to a PAYG pension system. Switching to a PAYG system (such as is attractive when switching from $r > g$ to $r = g$) is simple, since the stock of savings becomes available for current consumption. Switching back from PAYG to a funded system is hard, since one cohort has to give up consumption to rebuild the capital stock. We show that bubbly assets provide a useful instrument in this context. Suppose that r is temporarily low due to an investment slump. Then, the young do not want to invest all their savings in capital. Instead, they buy the bubbly asset. This raises the price of this asset. Hence, the elderly, who hold these assets, receive a windfall profit, which they spend on extra consumption. A reverse mechanism occurs during an investment boom. The variation in the price of the bubbly asset is therefore an instrument for shifting resources between consumption and investment, depending on the return on productive investment, or equivalently, it is an instrument for shifting aggregate consumption over time. Despite the risk of a future capital loss, buying bubbly assets is a rational strategy for the young.

The key contribution of the paper is to show the distinct role of fiscal policy in a world with unexpected shocks in the return to capital. In Tirole's balance growth world, bubbles and fiscal policy are perfect substitutes. Every additional dollar of sovereign debt reduces the value of the stock of bubbly assets by one dollar. This perfect substitutability no longer holds in an economy with unexpected shocks in r. Shifts between investment and consumption due to variations in r are equivalent to shifts in consumption between generations. One would expect that such shifts can only be implemented when enforced by the government. However, trade in bubbly assets is shown to be a substitute - albeit imperfect - for transfers enforced by the government. A bubbly equilibrium is therefore more efficient than the naive market equilibrium where all assets are priced according to the NPV of the expected future dividends. However, bubbly assets do not allow the implementation of the first-best allocation of resources to consumption and investment (the latter statement is contingent on the exact efficiency concept applied as the wealth of various cohorts has to be aggregated).

Even though bubbly assets are shown to further efficiency by shifting resources from investment to consumption during an investment slump and from consumption to investment during a boom, a simple fiscal policy rule is shown to be superior to trade in bubbly assets. The drawback of relying on trade in bubbly assets is that the elderly bear the full cost of adjustment. During an investment slump, the price they get for their holding of bubbly assets is high. Hence, they can consume more. The reverse holds during a boom. The full burden of the variability of consumption falls therefore on the elderly. Fiscal policy can share this burden between the young and the old. Consider a policy rule where the government commits to issuing a fixed amount of debt every period. It repays the debt from the previous period by receipts of the sale of new bonds this period. During an investment slump, the price of these bonds is high (i.e. the interest-rate is low). Hence, the government runs a surplus on its debt operations. This surplus is distributed among the young in the form of a temporary tax relief. The young save part of this tax relief for consumption during retirement, but another part will be spend on current consumption. The latter part contributes to the shift of resources from investment to consumption. This mechanism works the other way around during a boom. We enter a strange world in which bubbles implement *inter*generational transfers without enforcement by the government, while fiscal policy is a prerequisite for implementing *intra*generational transfers. Fiscal policy can therefore improve welfare. Under quite plausible parameter values, the level of sovereign debt should be set such as to eliminate bubbles entirely and to let sovereign debt absorb all the excess saving that gives rise to the dynamic inefficiency. Hence, no bubbles would emerge in this equilibrium.

We introduce the concepts of *ex post* and *ex ante* risk on the return on productive investments. *Ex post* risk is the standard type of stochastic uncertainty of which the realization is known only after the investment is made. It makes productive investment a risky endeavour. In contrast, the realization of *ex ante* risk is known at the moment that the investment is made. The part that is unknown is the *ex ante* risk on tomorrow's investment. This risk does not affect the return on today's productive investments, but it does affect the return on today's purchase of the bubbly asset, because tomorrow's price of the bubbly asset is negatively related to tomorrow's expected return on productive investment, which depends on tomorrow's *ex ante* risk.

Empirical research done after Tirole (1985) has cast doubt on the practical relevance of his argument. Andrew Abel, Gregory Mankiw, Lawrence Summers and Richard Zeckhauser (1987) showed that for an economy to be dynamically inefficient the capital sector must be a net sink: investment should exceed dividends. They showed that this condition is violated empirically by a wide margin. However, Francois Geerolf (2013) showed recently that when the criterion for the existence of rents is corrected for some factors (like natural resources), the economy might have been in a dynamic inefficient state frequently. This paper shows that when the economy switches back and forth between $r = g$ and $r > g$, the capital sector has positive outlays on average. Where Tirole's (1985) analysis yields the conclusion that $r = g$ is a prerequisite for the existence of rational bubbles, this paper leads to the conclusion that bubbles guarantee that $r > g$ "on average".

We extend the analysis one step further by allowing for risk aversion. Then, government debt also acts as an insurance device. Investing in sovereign debt when young offers a generation partial insurance to both the *ex ante* risk on buying bubbly assets and the *ex post* risk on investing in capital when they are old. The government charges an insurance premium on issuing these bonds that allows it to run a deficit "on average".

The condition of dynamic inefficiency has been argued not to correspond to what we usually associate with bubbly episodes. These episodes are characterized by high investment driven by waves of optimism, not by low investment, as in Tirole's model. Alberto Martin and Jaume Ventura (2012) analyze a world with distorted financial markets where bubbles enable the transfer of wealth from inefficient to efficient investors or where bubbles provide the collateral needed to support these transfers, see their 2014 paper and also Ricardo Caballero, Emmanuel Farhi and Mohamad Hammour (2006). However, not all bubbly episodes seem to support waves of high investment, consider, for example, the analysis of the hike in oil prices just before the demise of Lehman Brothers in 2008 by Caballero, Farhi, and Pierre Gourinchas (2008). Manuel

Santos and Michael Woodford (1997) worked out in their paper the conditions for the feasibility of rational bubbles.

This paper proceeds as follows. Section 2 considers the simplest economy that exhibits our mechanism. We use a convenient production function which allows an analytical solution by ruling out complementarity between labour and capital. The young save a fixed amount, which can be stored either in capital or in bubbly assets. The decision on how to store resources is the only margin of adjustment. Section 3 presents the core argument regarding fiscal policy. It extends the model of Section 2 by allowing for intertemporal substitution in consumption and by introducing the trade-off for a generation between consumption today and consumption tomorrow. This model has essentially two parameters, the average degree of dynamic inefficiency when all savings are invested in capital μ and the share η of lifetime wealth that is saved for retirement. These parameters characterize the essential mechanisms in our model. The higher the average degree of dynamic inefficiency, the higher must be sovereign debt to absorb excess saving. The lower η, the higher the share of the young in lifetime consumption and hence the larger the share of the burden cyclical adjustment in consumption that should be attributed to them. This can be done by fiscal policy only. Section 4 relaxes the assumption of risk-neutrality. Section 5 concludes.

2 The basic model

2.1 Core assumptions for all three models

The three models considered in this paper share a common set of assumptions. We consider an economy that is populated by overlapping generations living for two periods. During each period, a cohort of elderly dies, while a new young cohort enters the economy. In the first stage of their life, when young, this cohort works and receives labour income. In the second stage, when old, the cohort is retired and can only consume what is saved from the first stage. We apply the maximum convenience principle in modelling. Without loss of generality, we set the rates of population growth and technological progress equal to zero (hence: $g = 0$). The size of each cohort is normalized to unity. The young have two options for storing resources for consumption in the second stage of life: investing in vineyards or buying the bubbly asset. Their first option, to invest in vineyards, we refer to as productive investment. These investments are depleted in one period and yield a physical return. Since investments are fully depleted, investment is equal to the capital stock. Since $g = 0$, investing is dynamically efficient as long as one unit of investment yields a return of at least one unit of output. Alternatively, they can buy a bubbly asset, which we refer to as gold. Gold can neither be (re)produced nor become depleted. Its supply is normalized to unity. Holding gold does not enter the utility function. Say's law holds in this economy, expectations are rational, and markets clear and are perfectly competitive.

2.2 Assumptions for the basic model

In the basic model discussed in this section, the young save one unit of their income for consumption in the second stage of their life. The rest is consumed during the first stage. Since the income and the share of saving of the young are fixed by assumption, their consumption is also fixed. Hence, we focus entirely on the consumption of the old. All agents are risk-neutral. Hence, expected consumption of the old is a sufficient statistic for the utility of a generation.

Each member of a cohort owns a vineyard. When young, he chooses how much to invest in his vineyard. The relation between the input of capital in period t and output in period $t+1$ is given by a production function $f(\cdot)$:

$$\begin{aligned} f(k_t, u_t) &= \ln(k_t + u_t) + g(u_t), \qquad (1) \\ f_k(k_t, u_t) &= (k_t + u_t)^{-1}, \end{aligned}$$

for any $k_t + u_t > 0$; k_t is capital per worker, $0 \leq k_t \leq 1$ (since the capital stock can never exceed the available savings); $g(\cdot)$ is an arbitrary function, $g(\cdot) \geq 0$; u_t

is an i.i.d. technology shock with support $u_t \in [u^-, u^+]$, expectation $\mathrm{E}[u_t] = \mu$ and variance $\mathrm{Var}[u_t] = \sigma^2$.

This production function exhibits the standard feature of diminishing returns to capital with a positive first derivative and a negative second derivative, $f_k(k_t, u_t) > 0$ and $f_{kk}(k_t, u_t) < 0$, where the subscript k refers to the partial derivative with respect to k. The logaritmic functional form has the advantage that it yields a closed-form solution for our model. Other functional forms, like the standard $f(k_t, u_t) = (k_t + u_t)^\alpha$, $\alpha \in (0, 1)$, would yield similar results, but no closed form solutions. Our production function is non-standard as it does not allow complementarity between labour and capital. With complementarity, investment of the current generation youngsters would have a positive external effect on wages of the next generation. Leaving out this complementarity eliminates this effect. The technology shock u_t is additively capital augmenting: u_t is a perfect substitute for capital. Due to diminishing returns on capital, a higher u_t reduces the return on capital for a given level of k_t. A high value of u_t therefore leads to an investment slump, a low value to an investment boom.

The crucial feature of this economy is that u_t captures *ex ante* investment risk. *Ex ante* risk differs from the standard *ex post* risk in that its realization is known at the moment the young decide on how much to invest in their vineyard. *Ex post* risk is irrelevant in the current model with risk-neutrality. It will be introduced in Section 4, where we allow for risk aversion.

The function $g(u_t)$ plays no role in the analysis. It has been added here merely to show that the model can handle different types of technology shocks. Suppose $g(u_t) = 0$ for all u_t. Then output is an increasing function of u_t: an investment slump goes hand-in-hand with high levels of output. This is not the way we tend to think about investment slumps. We can deal with that by making $g(u_t)$ a declining function of u_t, such that investment and output are positively correlated. The only thing that matters for the analysis in this paper is the investment part of the story. Hence, for the sake of notational convenience, we set $g(u_t) = 0$, noting that we can generate any desired correlation between investment and output by a proper definition of $g(u_t)$.

Since $k_t + u_t > 0$ must hold for $\ln(k_t + u_t)$ to be defined, and since $k_t \in [0, 1]$, we must impose some constraints on the support of u_t. Since $k_t \leq 1$, a necessary condition is $u_t > -1$, implying $u^- > -1$. For all equilibria considered in this paper the condition $k_t > -u_t$ is satisfied. Since there is no growth in this economy, $g = 0$, the Aaron/Diamond condition for dynamic inefficiency $r < g$ reads $r < 0$. Since all capital is depleted in one period, the possibility of dynamic inefficiency requires that if all savings are invested in the worst state of nature, $k_t = 1$ and $u_t = u^+$, then the marginal return on capital must be less than one: $f_k(1, u^+) < 1$. Hence $u^+ > 0$. This motivates the following assumption on the upper and lower support of u_t:

$$\begin{aligned} u^- &> -1, \\ u^+ &> 0, \\ u^- &< \mu < u^+. \end{aligned} \tag{2}$$

The latter assumption follows immediately from the first two, since the mean must be an interior point of the support. The parameter μ can therefore be interpreted as the average share of savings that should not be invested in vineyards to maintain dynamic efficiency. If $\mu = 0$, there is on average sufficient investment demand to absorb the supply of savings.

Our model does not allow for autocorrelation in u_t. Allowing for autocorrelation is not difficult in principle, but it would complicate the derivations without affecting the main conclusions. We have not been explicit about the unit of time of our model, but the context of an overlapping generations model where people live for just two periods means that the appropriate unit of time is several decades. Then, the assumption of serial independence of subsequent values of u_t does not pose a serious problem.

Since Say's law holds, the sum of the investment of the young in their vineyards, k_t, and the consumption of the old, denoted c_t, must be equal to the sum of savings and the return on last period's investment minus current investment:

$$c_t = 1 + f(k_{t-1}, u_{t-1}) - k_t. \tag{3}$$

2.3 Characterization of the equilibrium

Let p_t be the price of gold in period t. In each period, members of the young generation must individually choose how much of their savings to invest in their vineyard, k_t. What is left is spent on gold. The young take this decision so as to maximize their expected return in the second period, which satisfies:

$$k_t = \arg\max_k f(k, u_t) + (1-k)\frac{\mathrm{E}_t[p_{t+1}]}{p_t},$$

where $\mathrm{E}_t[x]$ denotes the expectation of x conditional on the information available at time t. The first-order condition for the optimal portfolio composition reads:

$$f_k(k_t, u_t) = \frac{\mathrm{E}_t[p_{t+1}]}{p_t}. \tag{4}$$

The expected return on gold should be equal to the marginal productivity of capital. Market clearing on the market for gold requires that the young buy the entire stock of gold from the old. Since the young spend $1-k_t$ on gold and since the supply of gold is equal to unity, we have:

$$p_t = 1 - k_t. \tag{5}$$

An equilibrium is a solution for p_t and k_t that satisfies the first-order condition (4) and the market-clearing condition (5).

Proposition 1 *Equilibria for which $E_t[k_{t+1}]$ does not depend on t.*

1. *There exists an equilibrium where $k_t = 1$ and $p_t = 0$ for all realizations of u_t.*

2. *If:*

$$0 < \mu < \left(u^+\right)^{-1}, \tag{6}$$

then there exists a second equilibrium where:

$$\begin{aligned} k_t &= 1 - \frac{\mu}{1+\mu}(1+u_t), \\ p_t &= \frac{\mu}{1+\mu}(1+u_t). \end{aligned} \tag{7}$$

3. *If $\mu > 1/u^+$, then a similar equilibrium exists, but where investment is constrained by the non-negativity constraint $k_t \geq 0$ in some states of nature.*

The proof of Proposition 1 is instructive, as the two equilibria follow naturally as two distinct roots of a second-order polynomial in expected investment, $\mathrm{E}_t[k_{t+1}]$.

Proof. Substitution of condition (5) into equation (4) yields:

$$(1 - \mathrm{E}_t[k_{t+1}])(k_t + u_t) = 1 - k_t. \tag{8}$$

Taking expectations conditional on the information available on $t-1$ yields:

$$(1 - \mathrm{E}_{t-1}[k_{t+1}])(\mathrm{E}_{t-1}[k_t] + \mu) = 1 - \mathrm{E}_{t-1}[k_t]. \tag{9}$$

$\mathrm{E}_{t-1}[k_t]$ does not depend on the realization of any of the past or future shocks u_t. Hence, it is deterministic. Therefore $\mathrm{E}_{t-1}[k_{t+1}] = \mathrm{E}_t[k_{t+1}]$. Since we consider equilibria where $\mathrm{E}_t[k_{t+1}]$ does not depend on t, $\mathrm{E}_{t-1}[k_t] = \mathrm{E}_t[k_{t+1}]$ Using these results, equation (9) yields an expression for $\mathrm{E}_t[k_{t+1}]$:

$$0 = (1 - \mathrm{E}_t[k_{t+1}])(\mathrm{E}_t[k_{t+1}] + \mu - 1).$$

This equation has two solutions, $\mathrm{E}_t[k_{t+1}] = 1$ and $\mathrm{E}_t[k_{t+1}] = 1 - \mu$. The first equilibrium follows immediately from the first solution and equation (5). The second equilibrium follows from the substitution of $\mathrm{E}_t[k_{t+1}] = 1 - \mu$ in equation (8) and solving for k_t. Substitution in equation (5) yields an expression for p_t, proving equation (7). Consider this equation in detail. Since agents cannot be forced to sell their gold, the price of gold has to be positive in any state of nature. Hence: $\mu > 0$. Furthermore, the following must hold: $k_t \in [0, 1]$. The upper constraint is satisfied for all states of nature since $u^- > -1$ by equation (2). The lower constraint requires $\mu < 1/u^+$. This proves condition (6). ■

Proposition 1 discusses two possible equilibria. In the first equilibrium, the young invest all of their savings in vineyards, even when this is dynamically inefficient. Hence, the price of gold is zero. Nobody finds it attractive to buy gold, since its expected price - and therefore its current price - is zero.

If equation (6) holds, there is a second equilibrium where people find it attractive to buy gold. We refer to this equilibrium as the bubbly equilibrium. In this equilibrium, the young do not want to invest all their savings in vineyards when the return on this investment is low. Instead, they buy gold as an alternative store of value. They do so because the expected price of gold is positive. In equilibrium, the return on investment and the expected return on gold must be equal. When the return on capital is temporarily low, the price of gold is above its long-run equilibrium, since everybody wants to buy gold instead of investing in vineyards. Hence, the expected return on buying gold is also low, because people expect the price of gold to return to its long-run equilibrium, $\mathrm{E}_t[p_{t+1}] = 1 - \mathrm{E}_t[k_{t+1}]$, satisfying the return equivalence condition (4). The price of gold is a decreasing function of the investment in vineyards k_t. Since the expected price of gold is fixed, the variation in the expected return on gold is driven by variation in its current price.

The price of gold can never be zero in a bubbly equilibrium, since then the expected return would be infinite, which is inconsistent with the return equivalence condition (4). Hence, even in an extreme investment boom, $u_t = u^-$, not all saving is spent on productive investment, even though the return exceeds unity. This is also the reason why condition (6), $\mu > 0$, is stricter than condition (2), $u^+ > 0$. Condition (2) only guarantees that there are some states of nature where investing all savings in vineyards is inefficient. Condition (6) requires that investmenting all savings in vineyards is inefficient for the "average" state of nature. If not, the prospect of an investment slump would not be sufficiently severe for youngsters to buy gold even in the best state of nature.[1] In what follows, we assume condition (6) to apply.[2]

[1] Condition (6) also imposes $\mu < 1/u^+$. If this condition is violated, a bubbly equilibrium exists, but investment is bound by the non-negativity constraint $k_t \geq 0$ in some states of nature. Then all savings are spent on gold, leading to a messier description of the equilibrium. Hence, we omit it. Note that the upperbound $k_t \leq 1$ is never binding, since the young always spend some savings on gold.

[2] Since $\mu < u^+$, the condition $\mu < (u^+)^{-1}$ implies $\mu < 1$.

Ex ante risk in investment leads to *ex post* risk in the price of gold in a bubbly equilibrium: even though the current return on investment is known, the return on gold is uncertain, due to uncertainty about the future return on investment.

Proposition 1 focusses on equilibria where $\mathrm{E}_t[k_{t+1}]$ is constant over time. There are other equilibria, where $\mathrm{E}_t[k_{t+1}]$ varies over time, see e.g. Tirole (1985) for a discussion in the context of a deterministic equilibrium. There are sunspot equilibria and there are asymptotically bubbleless equilibria, where $r < g$. Then, the no arbitrage condition implies that the value of bubble declines gradually. A similar asymptotically bubbleless equilibrium can be expected to exist in this stochastic economy with random shocks to the product of capital. In that case, $\mathrm{E}_t[k_{t+1}]$ would gradually increase over time towards unity. This type of equilibrium is not essential for the argument on fiscal policy in the next section. Hence, I do not discuss them here.

We refer to the first equilibrium as the naive equilibrium, because it is unlikely that agents coordinate on this equilibrium when the bubbly equilibrium exists. As soon as we enter a bubbly world, where the expected price of gold is positive, $\mathrm{E}_t[p_{t+1}] > 0$, each individual agent is strictly worse off by not buying gold up till the point where the expected return on gold is equal to the return to investment in vineyards. Buying gold is therefore a rational decision even when everybody is aware that it is a bubbly asset. For this reason, an asymptotically bubbleless equilibrium is hard to justify. If we start to coordinate on believes that bubbly assets carry value in the first place, then why would we believe that the value of these assets will gradually decline? This holds *a fortiori* because bubbles satisfy a real demand for store of value during investment slumps. Thereby, they further efficiency, see Proposition 3 below. When bubbles vanish asymptotically, so would the efficiency gain.

Tirole (1985) shows that in a deterministic model, bubbles can only be sustained when $r = g$. Does this condition carry over to the stochastic model considered in this paper? Since the return on capital varies over time, we have to account for this variability. Hence, we investigate whether the expected return on capital satisfies this condition. Similarly, Abel et.al. (1987) show that permanent bubbles and permanent dynamic inefficiency would require the capital sector to be a net sink: investment should exceed capital outlays. The subsequent proposition shows that the expected return on capital is positive, $r > g$, and that the capital sector has on average net positive outlays.

Proposition 2 *The expected return on capital.*

1. $E\left[f_{kt}^{-1}\right] = 1$.

2. $E[f_{kt}] > 1$.

3. $E[f_{kt} \cdot k_t] - E[k_t] > 0$.

where $f_{kt} \equiv f_k\left(k_t, u_t\right)$.

Proof. The first statement follows directly from equation (4); the second from Jensen's inequality: $\mathrm{E}_t\left[f_{kt}^{-1}\right]\mathrm{E}_t\left[f_{kt}\right] > 1$; the third from

$$\mathrm{E}\left[f_{kt} \cdot k_t\right] - \mathrm{E}\left[k_t\right] = \left(\mathrm{E}\left[\frac{1+\mu}{1+u_t}\right] - 1\right)\mathrm{E}\left[k_t\right] + \mu\mathrm{Cov}\left[\frac{1}{1+u_t}, -u_t\right].$$

The first term is positive, by Jensen's inequality; the second term is also positive, since both stochasts depend negatively on u_t. ■

Proposition 2 shows that the results of Tirole and Abel et.al. do not apply "on average" in this economy. Rational bubbles exist even though the expected return on capital is positive and even though the capital sector has positive net outlays on average. The intuition behind the third result is that a technology shock is partially undone by lower investment. Hence, investment and the return on capital are positively correlated. This positive correlation causes the capital sector to have a positive net outlay on average. Hence, where Tirole concludes that bubbles can only be sustained when $r = g$, we conclude that the existence of bubbly assets guarantees that "on average" $r > g$.

2.4 Welfare comparison

An analysis of the implication of bubbles for welfare requires the definition of a proper criterion. The definition of welfare that we apply is expected utility of the young before the veil of ignorance about the *ex ante* risk u_t is lifted: that is, before the youngsters learn what prospects for productive investment their cohort faces. Since the consumption of the young is fixed, the first-best policy maximizes expected consumption of the elderly. Since we have no instruments to transfer wealth between periods other than investment in vineyards, the only decision we have to take is how much of the output generated in a particular period should be invested in vineyards and how much should consumed by the elderly. Since the former favours the young while the latter favours the elderly, any change in the allocation implies a transfer of wealth between generations.

Since the first-best investment rule $k(u_t)$ maximizes $\mathrm{E}[c_t]$, equation (3) implies

$$k(u_t) = \arg\max_k \left[1 + \ln(k + u_t) - k\right].$$

Note that both k_t and k_{t-1} enter equation (3), which depends on different realizations of the technology shock, u_t and u_{t-1} respectively. However, the rule $k(u)$ should apply likewise to both k_t and k_{t-1}. Since both terms enter additively, we can take expectations for each term separately and add up the expectations. This makes this formulation of the first-best policy applicable.[3] The first-order condition implies

$$k(u_t) = 1 - u_t. \tag{10}$$

Since $0 \leq k(u_t) \leq 1$, this condition applies unconstrained in all states of nature only if $0 \leq u^- < u^+ \leq 1$. This is a more stringent constraint than equations (2) and (6). Again, dealing with the truncations at $k(u) = 0$ and $k(u) = 1$ is straightforward in principle, but it messes up notation and provides no new insights. Hence, we assume that this more stringent condition holds. Furthermore,

[3] A more formal treatment would observe that the accumulated welfare of all future generations satisfies

$$\Sigma_{t=0}^{\infty}\mathrm{E}\left[c_t\right] = \Sigma_{t=0}^{\infty}\left(\mathrm{E}\left[\ln(k_{t-1} + u_{t-1})\right] + \mathrm{E}\left[1 - k_t\right]\right).$$

Taking the derivative of this expression with respect to k_t yields the same expression. The problem with this specification is that $\Sigma_{t=0}^{\infty}\mathrm{E}[c_t]$ does not converge. One could interpret this specification as the limiting case for the discount rate going to zero.

we assume that for any increasing function $h(\cdot)$, $\mathrm{E}[h(u_t)] = h(\mu) + \frac{1}{2}h''(\mu)\sigma^2$ and $\mathrm{Var}[h(u_t)] = h'(\mu)^2\sigma^2$.[4]

The consumption c_t that goes with this investment rule, see equation (3), depends on the investment opportunities of the young. This outcome is similar to a form of intergenerational insurance of the return on capital. If a generation faces a low marginal return on investment due to a high realization of u_t, then the current generation of the elderly absorbs the excess saving by consuming their windfall profit on bubbly assets. The young benefit indirectly from this obligation to hand over part of their savings to the current elderly without proper compensation, because the same rule that forces them to do so in this period, will apply also next period, when they are the potential beneficiaries. The first-best allocation can therefore be implemented by an optimal Rawlsian insurance policy that insures the young against the risk of a low return on productive investment agreed upon before the veil of ignorance is lifted. Insurance pays off even in this world with risk-neutral agents, since it allows agents to avoid having to make investments in vineyards when this is inefficient.

The bubbly equilibrium is a compromise between the naive market equilibrium and the first-best allocation: the coefficient on u_t in the expression for k_t is equal to zero in the naive equilibrium and equal to unity in the first-best equilibrium, while it is in between zero and unity in the bubbly equilibrium, $0 < \frac{\mu}{1+\mu} < 1$, see Proposition 1. Relative to the Rawlsian insurance policy, the bubbly equilibrium provides partial insurance. If a generation faces a low marginal return on capital, it invests less in its vineyards and spends more on bubbly assets. This reduces the volatility in the return on capital, but does not eliminate it. Full stabilization requires that k_t varies more. Then, the market-clearing condition $p_t = 1 - k_t$ implies that the price of gold should vary. Since the expected return on gold varies inversely to the price of gold, and since this return is equal to that on capital, this implies that the return on capital must

[4] This assumption says that $\mathrm{E}[h(u_t)]$ and $\mathrm{Var}[h(u_t)]$ are equal to the expressions derived from second-order Taylor expansions. The assumption is not crucial, but we apply this second-order expansion for comparative statics.

vary. It is therefore less volatile than in the naive market equilibrium, but more volatile than in the first-best allocation, where the return on capital is constant, $f_{kt} = 1$.

A peak in gold prices leads to a boost in the consumption of the elderly. Fluctuations in gold prices are therefore a means for adjusting the consumption of the elderly to the level of productive investment set by youngsters. When investment of the young is high, consumption of the elderly should be low, and the other way around. One would expect that this type of intergenerational transfer could not be implemented without enforcement by the government. However, bubbles are a substitute. The desire of the young to avoid unproductive investment by buying gold as an alternative store of value provides an enforcement mechanism for a partial implicit insurance contract. The market provides a second-best substitute for government-enforced intergenerational transfers in a bubbly equilibrium that is not available in the naive equilibrium.

Proposition 3 *The trade-off between expected welfare and its variability.*

1. *The naive equilibrium yields the highest mean level of investment; mean investment is the same in the first-best and the bubbly equilibrium.*

2. *First best yields the highest expected welfare and the naive equilibrium the lowest.*

3. *The ordering of the variability of welfare is the same.*

Proof. Remember that expected consumption of the elderly is a sufficient statistic for welfare. Hence, we can use the expressions for $\mathrm{E}[c_t]$ and $\mathrm{Var}[c_t]$ to prove the first two statements. Using equation (3), Proposition 1 and equation (10), one can derive the expressions presented in Table 1. Some simple calculations using these expressions prove the proposition. ■

Table 1 Expectation and variance of welfare and investment

equilibrium	naive	bubbly	first-best
c_t	$\ln(1+u_t)$	$\ln\left(\frac{1+u_t}{1+\mu}\right)+\frac{\mu}{1+\mu}(u_{t+1}+1)$	u_{t-1}
$\mathrm{E}[c_t]$	$\ln(1+\mu)-\frac{1}{2}\left(\frac{\sigma}{1+\mu}\right)^2$	$\mu-\frac{1}{2}\left(\frac{\sigma}{1+\mu}\right)^2$	μ
$\mathrm{Var}[c_t]$	$\frac{1}{(1+\mu)^2}\sigma^2$	$\frac{1+\mu^2}{(1+\mu)^2}\sigma^2$	σ^2
k_t	1	$\frac{1}{1+\mu}-\frac{\mu}{1+\mu}u_t$	$1-u_t$
$\mathrm{E}[k_t]$	1	$1-\mu$	$1-\mu$
$\mathrm{Var}[k_t]$	0	$\left(\frac{\mu}{1+\mu}\right)^2\sigma^2$	σ^2

There is a trade-off between expected welfare and its variability. Agents are risk-neutral, so the variability does not come at a price in this economy. This will change when we allow for risk aversion in Section 4. The sources of variation differ between equilibria. In the naive market equilibrium, the variation comes from the return on capital. In the first-best equilibrium, the variation comes from the investment in vineyards. The variation in the bubbly equilibrium is a mixture of both. Investment is the highest in the naive market equilibrium, since in that equilibrium agents have no alternative store of value. Remarkably, average investment is the same in the bubbly and the first-best equilibrium, though investment is more volatile in the first-best equilibrium. Hence, the bubbly equilibrium features overinvestment when the return on capital is low and underinvestment when the return is high.

3 Intertemporal substitution in consumption

3.1 Assumptions

The model of Section 2 constrained the problem to the allocation of current saving to either investment in vineyards (to the benefit of the young) or consumption (to the benefit of the elderly). In practice, consumption can be transferred between stages of life. This section introduces a trade-off for the young between consumption now and saving for future consumption. To fix ideas, suppose that agents are characterized by Epstein-Zin (1989) preferences with risk-neutrality within each period and Cobb Douglas preferences for intertemporal substitution:

$$U = (1-\eta)\ln \mathrm{E}\left[c_t^y\right] + \eta \ln \mathrm{E}\left[c_{t+1}\right], \tag{11}$$

where c_t^y is the consumption of the young cohort entering the economy at t and c_{t+1} is their consumption when they have grown old at $t+1$. The parameter η is the budget share of consumption in the second period, $0 < \eta < 1$. Epstein-Zin preferences decouple the rate of intertemporal substitution from the degree of risk aversion. This section maintains the assumption of risk-neutrality.

The Cobb Douglas structure for intertemporal substitution implies that income and substitution effects cancel, so that variations in the (expected) return on capital do not affect the budget share that the young set apart for future consumption. Total labour income earned in the first period of life is conveniently assumed to be equal to η^{-1}, so that savings are equal to unity in a market equilibrium, in both the naive and the bubbly equilibrium, as in the previous section.

We can achieve an analytical solution only by using a simplification compared to most of the literature. In an economy with constant returns to scale, the investment in capital by the previous generation yields a positive externality to the wage rate faced by the next generation. This externality is equal to the difference between the marginal and the intra-marginal return to capital. It introduces a source of persistence: higher investment today yields higher wages and hence higher savings next period. Our production function rules out complementarity between labour and capital, but this raises the issue what happens to the difference between the marginal and the intra-marginal return to capital. Here, we introduce a separate class of rentiers, who own the vineyards and consume all the rents derived from their property, but who play no further role in the economy. We assume that there is no market for vineyards.

Capital productivity is modelled in the same way as in the previous section, except that vineyards are owned not by the population at large, but by a separate class of rentiers. The difference between the marginal and the intra-marginal return, $f(k_t, u_t) - f_k(k_t, u_t)\, k_t$, is the income of the rentier class, which plays no further role in the analysis.

3.2 Fiscal policy

Since income and substitution effects cancel in the intertemporal trade-off of youngsters, the naive and the bubbly equilibrium are exactly the same as in the previous section, apart from the difference due to the introduction of a separate rentier class. However, the extension of the model with intertemporal substitution in consumption allows us to study the effect of fiscal policy. We consider a simple policy rule, where the government issues bonds at time t that pay back one unit of consumption per bond at time $t+1$. The government commits to issue b bonds of this type each period. It sells them at a price q_t, which differs between periods. Hence, the interest rate is equal to $q_t^{-1}-1$. Each period, the government has to repay its debt b, but it receives bq_t from new bond issuance. The difference between the two is covered by a tax z_t on labour income (or: subsidy, if $q_t > 1$), which satisfies

$$z_t = b\left(1-q_t\right). \tag{12}$$

Lifetime wealth is equal to gross labour income η^{-1} minus the tax on labour income. Hence, agents consume an amount $\frac{1-\eta}{\eta}\left(1-\eta z_t\right)$ when young and save an amount $1-\eta z_t$ for future consumption. Hence, expected consumption of the elderly at time $t+1$ evaluated at time t satisfies

$$\begin{aligned} \mathrm{E}_t\left[c_{t+1}\right] &= \left(1-\eta z_t\right)\mathrm{E}_t\left[R_t\left(s,g\right)+1\right], \\ R_t\left(s,g\right) &\equiv q_t^{-1}+s\left(f_{kt}-q_t^{-1}\right)+g\left(p_t^{-1}p_{t+1}-q_t^{-1}\right)-1, \end{aligned} \tag{13}$$

where s and g are the shares of savings held in investment and gold, respectively. Note that $R_t\left(g,s\right) \neq \mathrm{E}_t\left[R_t\left(g,s\right)\right]$, since R_t depends on u_{t+1}. The term $1-\eta z_t$ represents the savings set apart for consumption in the second period; R_t+1 measures the return on that savings. The young choose the composition of their portfolio as to maximize $R_t\left(g,s\right)$:

$$g_t, s_t = \arg\max_{g,s} \mathrm{E}_t\left[R_t\left(g,s\right)+1\right]. \tag{14}$$

The first-order conditions of this problem require the expected return on the three available assets to be equal:

$$f_k(k_t + u_t) = q_t^{-1} = p_t^{-1}\mathrm{E}_t[p_{t+1}]. \tag{15}$$

Market clearing requires investment k_t to be equal to its share s_t in total savings, and likewise for bonds and gold, implying

$$\begin{aligned} k_t &= (1-\eta z_t)\, s_t, \\ bq_t &= (1-\eta z_t)(1-g_t-s_t), \\ p_t &= (1-\eta z_t)\, g_t = 1-\beta(\eta_0+q_t)-k_t, \end{aligned} \tag{16}$$

where equation (12) is substituted for z_t and where $\beta \equiv (1-\eta)\, b$ and $\eta_0 \equiv \eta/(1-\eta)$. An equilibrium is a quintuple g_t, s_t, p_t, k_t, and q_t that solves equation (15) and the market-clearing conditions (16).

Proposition 4 *The effect of fiscal policy in the bubbly equilibrium.*

1. *A bubbly equilibrium exists when* $b \leq \mu$.

2. *The expected price of gold* p_t *is* $\mu - b$; *the expected price of bonds* q_t *is unity.*

3. *The expected level of investment* k_t *is* $1-\mu$ *and hence does not depend on* b.

4. *Investment and the price of gold are less variable for a higher* b, *the prices of gold and bonds are more volatile for a higher* b.[5]

5. *Expected utility before lifting the veil of ignorance about* u_t *reaches a maximum for* $b = \min\left[\frac{1-\mu}{\eta}, \mu\right]$.

Proof. see Appendix. ■

Bubbles cease to exist when $b \geq \mu$, see Statement 1. Since the expected level of investment is $1-\mu$, the expected demand for stores of value is equal

[5] The variability of x_t refers to $\mathrm{Var}[x_t]$, while its volatility refers to $\mathrm{Var}[\ln x_t]$.

to μ. Hence, the demand for gold vanishes when the government absorbs the excess supply of saving by issuing bonds. Furthermore, the price of gold must be positive even for the highest investment demand, $u_t = u^-$, yielding a constraint on u^-.

Statement 2 says that any savings absorbed by fiscal policy leads to an equal reduction in the expected price of gold. Statement 3 states that the average level of investment in a bubbly equilibrium is equal to the first-best, irrespective of the level of b, compare this with Statement 1 of Proposition 3. However, investment is less sensitive to shocks to the return on investment, the more so the higher b, see Statement 4. Hence, this simple fiscal policy drives investment away from first-best compared to the bubbly equilibrium without fiscal policy. The reason is that bonds have a fixed pay out, while the future price of gold varies according to the state of investment demand. Hence, bubbly assets are better suited to accommodate variation in investment demand.

Nevertheless, this simple policy improves welfare compared to the equilibrium without fiscal policy, see Statement 5. The reason is that fiscal policy allows transferring consumption between the two stages of life. This cannot be achieved in a bubbly equilibrium without fiscal policy, since the young always consume a share $1 - \eta$ of their lifetime wealth. Hence, when lifetime wealth is constant, so will be consumption in the first stage of life. Only the elderly adjust their consumption in response to an investment boom or a slump, by the same mechanism as in the economy without intertemporal substitution. When the government conducts fiscal policy, the young share in the absorption of shocks in total consumption. This is achieved by changing the lifetime wealth of the young. When investment is low due to a low return on capital, demand for government bonds is high, leading to a high price of bonds and hence a low interest rate. Hence, taxes z_t will be negative, which raises lifetime wealth and hence consumption of the young. This increase does not affect their consumption when retired, due to the fall in the return on bonds. Stated differently, fiscal policy uses the income effect of negative taxes to boost current consumption during an investment slump and the substitution effect of a low interest rate to offset the income effect in the second stage of life. Fiscal policy is therefore most effective when the share of consumption during retirement η is small. The smaller η, the less attractive it is to let all shocks in consumption be absorbed by the elderly. Hence, there is a larger role for fiscal policy to shift consumption between the stages of life. Contrary to the standard view of fiscal policy dealing with *inter*generational transfers, fiscal policy implements *intra*generational transfers of consumption in this world, while trade in bubbly assets implements the *inter*generational transfers. When $\frac{1-\mu}{\mu} < \eta$ (that is, for a high average level of inefficiency μ), welfare is maximized by some combination of both institutions. If the average level of inefficiency is low, welfare is maximized by putting the full burden of adjustment on the young. Since η (the share of the consumption when retired in lifetime wealth) is of the order of magnitude of 0.25, the average level of inefficiency must be quite high for the former to be optimal. Hence, it is quite likely that fiscal policy alone should do the job in this world. Sovereign debt would be sufficiently high in that case to prevent bubbles emerging. Note also that at the critical transition $\frac{1-\mu}{\mu} = \eta$, the relation between the optimal level of b and μ changes sign. For lower levels of inefficiency, a higher value of μ increases the optimal value of b (which is μ, in that case). However, as long as $\frac{1-\mu}{\mu} < \eta$, a greater degree of expected dynamic inefficiency, μ going up, should lead to a higher level of sovereign debt.

The fiscal policy considered here applies an income tax to cover the deficits or surpluses from the government's debt operations. Hence, only the young pay taxes or receive subsidies. One could generalize this policy by allowing for a combination of income and consumption taxes, thereby spreading the tax burden between the young and the elderly. A proper combination of income and consumption taxes could implement the same equilibrium as is implemented by a combination of income taxes and bubbly assets. Such a combination of income and consumption taxes is a substitute for bubbly assets.

Even when such a combination of income and consumption taxes is available, the type of fiscal policy considered here is quite simple, since we constrain b to be constant over time. Would more complicated policy rules allow for a further improvement of welfare? The answer is definitely yes. One can show that a first-best allocation would require more complicated investment and consumption rules, which depend not only on u_t, but also on k_{t-1} and u_{t-1}. A more complicated and activist fiscal policy would therefore improve welfare beyond the constrained optimum considered in Proposition 4. We do not present this first-best allocation here, since it has no analytical solution and is therefore hard to characterize, while it contributes little to understanding the relevant mechanisms.

Statements 2 and 4 say that fiscal policy stabilizes financial markets in the sense that it reduces both the average price of gold and its variability in absolute terms; in relative terms, the variability of the price of gold increases. There is less demand for gold as a store of value since sovereign debt serves as a substitute. The only difference between gold and government bonds is that the return on gold is risky (since its future price depends on the future return on capital), while the return on bonds is fixed. However, since agents are risk-neutral in this economy, this difference is irrelevant here. This will change when we introduce risk aversion.

4 Risk aversion and the risk-free rate

4.1 Assumptions

This section relaxes the assumption of risk-neutrality, while maintaining the Cobb Douglas structure for intertemporal substitution:

$$U = (1-\eta)\ln \mathrm{E}\left[(c_t^y)^{1-\gamma}\right]^{1/(1-\gamma)} + \eta \ln \mathrm{E}\left[c_{t+1}^{1-\gamma}\right]^{1/(1-\gamma)}. \tag{17}$$

The parameter γ is the degree of relative risk aversion. For $\gamma = 1$, the utility function simplifies to

$$U = \mathrm{E}\left[(1-\eta)\ln c_t^y + \eta \ln c_{t+1}\right].$$

In that case, we are back in the standard expected utility framework.

Thus far, the riskiness of the investment in vineyards has been irrelevant. Under risk-neutrality, the only thing that mattered was *ex ante* risk. Since the realization of this factor is known at the moment of investment, the investment itself is risk-free in an economy with only this type of risk. However, investment is risky in reality. In an economy with risk aversion, this uncertainty should be taken into account. We therefore extend the production function with an additional random variable accounting for *ex post* risk:

$$f\left(k_t, u_t, v_{t+1}\right) = \left(1 + v_{t+1}\right) \ln\left(k_t + u_t\right),$$

where v_t is an i.i.d. technology shock with $\mathrm{E}[v_t] = 0$,$\mathrm{Var}[v_t] = \chi^2$ and $\mathrm{Cov}[u_t, v_{t+1}] = \mathrm{Cov}[u_t, v_t] = 0$. v_{t+1} and u_t are independent by construction: v_{t+1} captures the new information that is coming in at $t+1$. If that information were to be correlated to u_t, u_t would contain information about the future value of v_{t+1}, and hence v_{t+1} would not be news.[6] The more substantive assumption is that u_t and v_t are uncorrelated. One would surmise that the expected return on future investment is correlated to the realized return on current investment. Allowing for this correlation is straightforward in principle, but would mess up subsequent derivations. Hence, it is ruled out by assumption. The production function implies

$$\begin{aligned} f_{kt} &= \frac{1 + v_{t+1}}{k_t + u_t}, \qquad (18) \\ \mathrm{E}_t\left[f_{kt}\right] &= \frac{1}{k_t + u_t}, \\ \mathrm{Var}_t\left[f_{kt}\right] &= \left(\frac{\chi}{k_t + u_t}\right)^2, \end{aligned}$$

where $f_{kt} \equiv f_k\left(k_t, u_t, v_{t+1}\right)$: the marginal productivity of capital. Note that $f_{kt} \neq \mathrm{E}_t\left[f_{kt}\right]$ due to *ex post* investment risk v_{t+1}.

Fiscal policy is the same as in the previous section.

[6] Strictly, this argument would apply only when u_t and v_t would enter additively: $f(\cdot) = \ln\left(k_t + u_t + v_{t+1}\right)$. Up to a second-order term, this specification is identical to the specification in the text. The latter specification is somewhat more convenient in the subsequent analysis.

4.2 Characterization of the equilibrium

As in the model with risk-neutrality, youngsters save $1-\eta z_t$ for consumption in the second stage of life. Hence, equation (13) for c_{t+1} applies. Agents choose g_t and s_t as to maximize $\mathrm{E}_t\left[c_{t+1}^{1-\gamma}\right]^{1/(1-\gamma)}$. Since we can factor out $\mathrm{E}_t\left[(1-\eta z_t)^{\gamma}\right] = (1-\eta z_t)^{\gamma}$ and since only R_t depends on g_t and s_t, the problem can be written as

$$g_t, s_t = \arg\max_{g,s} \mathrm{E}_t\left[\left[R_t\left(g,s\right)+1\right]^{1-\gamma}\right]^{1/(1-\gamma)}. \tag{19}$$

An equilibrium is a quintuple g_t, s_t, p_t, k_t, and q_t that solves equation (19) and the market-clearing conditions (16).

Since a full characterization of this equilibrium is too difficult a task, due to the non-linearity of equation (19) (compare the linearity of equation 14), we take a step back. We approximate the optimal portfolio for small deviations of u_t and v_t from their expected value. In particular, we assume

$$\begin{aligned} \sigma &= h \cdot \sigma_0, \qquad (20)\\ \chi &= h \cdot \chi_0, \end{aligned}$$

where we consider the equilibrium for the limiting case of the standard deviations of u_t and v_t being small: $\lim h \to 0$. Hence: $u_t - \mu = O\left(h\right)$ and $v_t = O\left(h\right)$. The following proposition allows a Taylor approximation of the market returns to the assets.

Proposition 5 *The return equivalence conditions in the bubbly equilibrium read*

$$\begin{aligned} q_t - k_t - u_t &\cong \psi_k, \qquad (21)\\ q_t - 1 - \pi\left(u_t - \mu\right) &\cong \psi_p, \end{aligned}$$

where the symbol $\cong$ *implies that terms of* $O\left(h^3\right)$ *are ignored and where* $\psi_k \equiv \gamma\left(1-\mu\right)\chi^2$, $\psi_p \equiv \gamma\left(\mu - b\right)\pi^2\sigma^2$, *and* $\pi \equiv \left(1-\eta b+\mu\right)^{-1}$; $d\pi/db > 0$ *and* $d\psi_p/db < 0$.

Proof. see Appendix. ■

The approximation applied in Proposition 5 accounts for the effect of realizations of the *ex ante* and *ex post* risk factors u_t and v_t on the composition of the asset portfolio of the young, and hence for the effect of these risk factors on asset prices. This effect is of order h. The approximation also accounts for the effects of the average portfolio composition on risk premia and utility, which are effects of order h^2. The approximation ignores the effect of deviations from the average portfolio on utility due to the realization of u_t and v_t. Since deviations are of order h and have a utility cost of order h^2, this effect is of order h^3. The intuition is that the average portfolio balances the utility cost of these risks on average, so that deviations have a higher-order effect.

Equation (21) generalizes the return equivalence conditions (15) for the case of risk aversion. The inverse risk-free rate q_t minus the inverse of expected rate of return on risky assets (either productive investments or gold) is equal to the risk premia on holding either of these assets (ψ_k and ψ_p respectively). Since the expected rate of return on gold is equal to the growth rate of the economy -which is zero- the risk premium on holding gold implies that the expected return on bonds is negative. This negative return is the premium for insurance against future *ex ante* risk u_t. While the risk premium on investment ψ_k does not depend on the fiscal policy parameter b, the risk premium on the bubbly asset ψ_p is decreasing in b. When the government absorbs a larger share of savings, it has to pay a higher interest rate on its bonds (or equivalently, the price of bonds will be lower).

An approximate equilibrium is the solution for k_t, p_t, and q_t to the asset return equations (21) and the market-clearing condition (16) for p_t. The next proposition characterizes the equilibrium.

Proposition 6 *Risk aversion and fiscal policy.*

1. *A bubbly equilibrium exists when* $b \leq \mu$.

2. *This equilibrium satisfies:*

$$\begin{aligned} k_t &\cong 1-\mu-\pi\left(\mu-\eta b\right)\left(u_t-\mu\right)+\psi_p-\psi_k, \\ p_t &\cong \mu-b+\pi\left(\mu-b\right)\left(u_t-\mu\right)+\psi_k-\left(1+\beta\right)\psi_p, \\ q_t &\cong 1+\pi\left(u_t-\mu\right)+\psi_p. \end{aligned} \tag{22}$$

3. *The expected level of investment is lower for higher b.*

4. *The expected price of gold and of bonds is lower for higher b.*

5. *Investment and the price of gold are less variable for a higher b; the prices of gold and bonds are more volatile for a higher b.*

6. *The higher the degree of risk aversion γ, the stronger the effect of fiscal policy on utility when measured at $b=0$: $\frac{dU}{db}_{|b=0}$ is increasing in γ.*

Proof. see Appendix. ■

For $\gamma = 0$ this approximate equilibrium is identical to the equilibrium considered in Proposition 4, compare equation (22) to equation (23) in the Appendix. The expected value of investment deviates from that in the risk-neutral economy, depending on which has the higher risk premium: gold or investment in vineyards. When *ex ante* risk dominates, risk aversion leads to higher investment, since the alternative of holding gold is more risky. Bonds and gold are close substitutes, apart from the fact that gold is a risky asset due to next period's *ex ante* risk. Fiscal policy has a negative effect on the price of gold: when sovereign debt acts as an alternative store of value, the average demand for gold goes down. However, this initial effect is partly offset by a reduction in the risk premium ψ_p on holding gold, since fiscal policy stabilizes the price of gold.

The introduction of a simple fiscal policy dampens the variability of investment and the price of gold. However, fiscal policy is a less effective means of shifting resources between investment and consumption than trade in bubbly assets. Hence, investment is less responsive to variations in its return. The price of gold is less variable because government bonds are available as an alternative store of value. Risk aversion increases the marginal effect of fiscal policy on utility because it provides better insurance for the young, since part of the adjustment in the balance between consumption and investment is spread between the two stages of life.

5 Conclusion

This paper analyzed a world where bubbles are a means for implementing intergenerational transfers to accommodate temporary fluctuations in the return on capital. Bubbly assets serve as an alternative store of value in the presence of dynamic inefficiency. Despite these temporary episodes of dynamic inefficiency, capital is productive on average, in the sense that average outlays of the capital sector exceed inflow, a criterion very similar to that derived by Abel et.al. (1987). In this world, a simple fiscal policy stabilizes the economy where the government issues a fixed quantity bonds with a fixed future pay out. These bonds serve as an alternative store of value during investment slumps, thereby providing an instrument for *intra*generational transfers and reducing the price of bubbly assets. Variation in the price of these bonds provides a means for adjusting consumption to investment demand. Remarkably, trade in bubbly assets shifts resources between investment and consumption by *inter*generational transfers, while fiscal policy does this by *intra*generational transfers, shifting resources over the lifecycle between current and future consumption. These results counter the conventional wisdom that only the government can enforce *inter*generational transfers and that fiscal policy is the means to implement these transfers. For reasonable parameter values, this simple model predicts that welfare is maximized by letting sovereign debt absorb on average all excess saving, thereby preventing the emergence of bubbles.

Our analysis applies a Cobb Douglas intertemporal utility function, implying an elasticity of intertemporal substitution equal to one. When the elasticity of intertemporal substitution is less than unity, as is usually found empirically, the effect of investment slumps becomes even stronger. A fall in the return on capital will then lead to an increase in the budget share that the young set apart for future consumption, putting greater strains on the ability of the capital market to absorb these savings. Then, the fiscal policy has to play an even larger role in spreading the consequences of fluctuations in the demand for capital among cohorts.

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Appendix Proofs

Proof of proposition 4

The first equality of equation (15) can be written as: $q_t = k_t + u_t$. Substitution into the market-clearing condition (16) for p_t and taking expectations yields

$$\begin{aligned} p_t &= 1 - \beta(\eta_0 + u_t) - (1+\beta) k_t, \\ \mathrm{E}_t[p_{t+1}] &= 1 - \beta(\eta_0 + \mu) - (1+\beta) \mathrm{E}_t[k_{t+1}]. \end{aligned}$$

Substitution of these expression in the second equality of equation (15) yields

$$(k_t + u_t)[1 - \beta(\eta_0 + \mu) - (1+\beta)\mathrm{E}_t[k_{t+1}]] = 1 - \beta(\eta_0 + u_t) - (1+\beta) k_t.$$

This equation shows that k_t depends on u_t and $\mathrm{E}_t[k_{t+1}]$ only. Hence, equation (**??**) applies. Taking expectation in the final equation yields:

$$0 = (1 - \beta(\eta_0 + \mu) - (1+\beta)\mathrm{E}[k_t])(\mathrm{E}[k_t] + \mu - 1).$$

This equation has two solutions. The solution that sets the first factor equal to zero corresponds to the naive equilibrium. We focus on the second solution corresponding to the bubbly equilibrium:

$$\mathrm{E}[k_t] = 1 - \mu.$$

Some calculation yields expressions for investment and the prices of bonds and gold:

$$\begin{aligned} k_t &= \frac{\beta_0 - (\mu - \eta b) u_t}{\beta_0 + \mu}, \\ q_t &= \frac{\beta_0 + u_t}{\beta_0 + \mu}, \\ p_t &= (\mu - b)\frac{\beta_0 + u_t}{\beta_0 + \mu}, \end{aligned} \tag{23}$$

where $\beta_0 \equiv 1 - \eta b$; p_t should be positive for any state of nature for an equilibrium to exist, proving statement 1. The expected prices of bonds and gold follow immediately:

$$\mathrm{E}[k_t] = 1 - \mu, \mathrm{E}[q_t] = 1, \mathrm{E}[p_t] = \mu - b,$$

proving statements 2 and 3. Statement 4 follows from equation (23).

Consumption for the young and the elderly satisfies the following:

$$\begin{aligned}
\mathrm{E}\left[c_t^y\right] &= \mathrm{E}\left[\eta_0^{-1}\left(1-\eta z_t\right)\right] = \eta_0^{-1}, \\
\mathrm{E}_t\left[c_{t+1}\right] &= \frac{1-\eta z_t}{q_t} = \beta_0 q_t^{-1} + \eta b \\
&= 1 + \beta_0 \left(\frac{dq_t}{du_t}\right)^2 \sigma^2 = 1 + \frac{\beta_0}{\left(\beta_0+\mu\right)^2}\sigma^2.
\end{aligned}$$

Substitution in expected welfare yields:

$$U = -\left(1-\eta\right)\ln\eta_0 + \eta\ln\left(1+\frac{\beta_0}{\left(\beta_0+\mu\right)^2}\sigma^2\right). \tag{24}$$

The first-order condition for the optimal value of b implies

$$\Rightarrow \frac{d}{db}\left(\frac{\beta_0}{\left(\beta_0+\mu\right)^2}\right) = \eta\frac{\beta_0-\mu}{\left(\beta_0+\mu\right)^3} = 0.$$

$b = \frac{1-\mu}{\eta}$ solves this equation. However, $b \leq \mu$ for a bubbly equilibrium to exist, see statement 1. Hence, the optimal value is $b = \min\left[\frac{1-\mu}{\eta}, \mu\right]$, proving statement 5.■

Proof of proposition 5

Conjecture 7 *There is an equilibrium with the following properties:*

1. $q_t - 1 = O\left(h\right)$

   $\Rightarrow q_t^{-1} - 1 = O\left(h\right), z_t = b\left(1-q_t\right) = O\left(h\right)$; *see equation (12).*

2. $f_{kt} - 1 = O\left(h\right) \Rightarrow f_{kt}^{-1} - 1 = O\left(h\right)$

   $\Rightarrow E_t\left[f_{kt}\right]^{-1} - 1 = k_t + u_t - 1 = O\left(h\right)$; *see equation (18)*$\Rightarrow E[k_t] = 1-\mu+O\left(h\right)$.

3. $p_t E_t\left[p_{t+1}\right]^{-1} \cong 1 + \pi\left(u_t-\mu\right)$, *where* $\pi = \left(\beta_0+\mu\right)^{-1}$

   $\Rightarrow Var_t\left[p_t E_t\left[p_{t+1}\right]^{-1}\right] \cong Var_t\left[p_t^{-1}E_t\left[p_{t+1}\right]\right] \cong \pi^2\sigma^2$

The strategy of the proof is to apply the conjecture to derive the equilibrium and then show that this equilibrium satisfies the conjecture.

Lemma 8 *Let x be a random variable with $E[x]=1+\mu$ and $Var[x]=\sigma^2$ with $\mu=O(h)$ and $\sigma=O(h)$. Then*

$$E\left[x^{1-\gamma}\right]^{1/(1-\gamma)}\cong 1+\mu-\frac{1}{2}\gamma\left(\sigma^2+\mu^2\right).$$

Proof. Define $m\equiv\ln(1+\mu)$ and $\overline{\gamma}=1-\gamma$. We have

$$\begin{aligned}
m &\cong \mu-\frac{1}{2}\mu^2=O(h),\\
\mu &\cong m+\frac{1}{2}m^2,\\
\mathrm{E}[\ln x] &\cong m-\frac{1}{2}\sigma^2,\\
\mathrm{Var}[\ln x] &\cong \sigma^2.
\end{aligned}$$

Hence

$$\begin{aligned}
\mathrm{E}\left[e^{\overline{\gamma}\ln x}\right]^{1/\overline{\gamma}} &= \left[e^{\overline{\gamma}\mathrm{E}[\ln x]}+\frac{1}{2}\overline{\gamma}^2e^{2\overline{\gamma}\mathrm{E}[\ln x]}\mathrm{Var}[\ln x]+O\left(h^3\right)\right]^{1/\overline{\gamma}}\\
&= \left[1+\overline{\gamma}\mathrm{E}[\ln x]+\frac{1}{2}\overline{\gamma}^2\mathrm{E}[\ln x]^2+\frac{1}{2}\overline{\gamma}^2\mathrm{Var}[\ln x]+O\left(h^3\right)\right]^{1/\overline{\gamma}}\\
&= \left[1+\overline{\gamma}m+\frac{1}{2}\overline{\gamma}^2m^2-\frac{1}{2}\overline{\gamma}\gamma\sigma^2+O\left(h^3\right)\right]^{1/\overline{\gamma}}\\
&\cong 1+m+\frac{1}{2}\overline{\gamma}m^2-\frac{1}{2}\gamma\sigma^2\cong 1+\mu-\frac{1}{2}\gamma\left(\mu^2+\sigma^2\right).
\end{aligned}$$

■

The conjecture and assumption (20) allow a Taylor expansion of equation (19):

$$g_t,s_t\cong\arg\max_{g,s}\left(\mathrm{E}_t\left[R_t(g,s)\right]-\frac{1}{2}\gamma\mathrm{E}_t\left[R_t(g,s)\right]^2-\frac{1}{2}\gamma\mathrm{Var}_t\left[R_t(g,s)\right]\right).$$

By equation (13), we have:

$$\begin{aligned}
R_t(g,s) &= (1-s-g)\,q_t^{-1}+sf_{kt}+gp_t^{-1}p_{t+1}-1, \qquad (25)\\
\mathrm{E}_t\left[R_t(g,s)\right] &= (1-s-g)\,q_t^{-1}+s\left(k_t+u_t\right)^{-1}+gp_t^{-1}\mathrm{E}_t\left[p_{t+1}\right]-1=O(h),\\
\mathrm{Var}_t\left[R_t(g,s)\right] &\cong s^2\chi^2+g^2\pi^2\sigma^2.
\end{aligned}$$

In the second line, we use equation (18) for $\mathrm{E}_t[f_{kt}]$ in the first equality and the Conjecture in the second equality. In the third line, we use equation (18) for

$\mathrm{Var}_t\left[f_{kt}\right]$ (noting that $k_t+u_t=1+O\left(h\right)$ by the Conjecture) and the Conjecture for $\mathrm{Var}_t\left[p_{t+1}\right]$. Note that both χ^2 and σ^2 are $O\left(h^2\right)$ so that we can ignore any higher-order terms.

The first-order conditions for g_t and s_t read as follows

$$\begin{aligned} \gamma\chi^2 s_t &\cong \left(1-\gamma\mathrm{E}_t\left[R_t\right]\right)\left(\mathrm{E}_t\left[f_{kt}\right]-q_t^{-1}\right), \\ \gamma\sigma^2\pi^2 g_t &\cong \left(1-\gamma\mathrm{E}_t\left[R_t\right]\right)\left(p_t^{-1}\mathrm{E}_t\left[p_{t+1}\right]-q_t^{-1}\right). \end{aligned} \tag{26}$$

where $R_t \equiv R_t\left(g_t, s_t\right)$. By the Conjecture, the market-clearing conditions (16) can be written as

$$\begin{aligned} s_t &= \left(1-\eta z_t\right)^{-1} k_t = \mathrm{E}\left[k_t\right]+O\left(h\right), \\ g_t &= \left(1-\eta z_t\right)^{-1} p_t = \mathrm{E}\left[p_t\right]+O\left(h\right). \end{aligned} \tag{27}$$

Substitution of equation (27) into equation (26), multiplying the result by $1-\gamma\mathrm{E}_t\left[R_t\right]$, and observing that χ^2 and σ^2 are $O\left(h^2\right)$ and $\mathrm{E}_t\left[R_t\right]=O\left(h\right)$ (see equation 25) yields

$$\begin{aligned} \gamma\chi^2\mathrm{E}\left[k_t\right] &\cong \psi_k \cong \mathrm{E}_t\left[f_{kt}\right]-q_t^{-1}, \\ \gamma\pi^2\sigma^2\mathrm{E}\left[p_t\right] &\cong \psi_p \cong p_t^{-1}\mathrm{E}_t\left[p_{t+1}\right]-q_t^{-1}. \end{aligned} \tag{28}$$

Dividing the first equation by $\mathrm{E}_t\left[f_{kt}\right]$ and the second by $p_t^{-1}\mathrm{E}_t\left[p_{t+1}\right]$, multiplying both equations by q_t, and observing that $\mathrm{E}_t\left[f_{kt}\right]^{-1}=1+O\left(h\right)$, $p_t\mathrm{E}_t\left[p_{t+1}\right]^{-1}=1+O\left(h\right)$, and $q_t^{-1}=1+O\left(h\right)$, see the Conjecture, and using that both χ^2 and σ^2 are $O\left(h^2\right)$, yields equation (21).

Equations (16) (that for p_t) and (21) is a system of three linear equations with three unknowns: k_t, p_t, and q_t. The solution to this system reads as follows:

$$\begin{aligned} k_t &\cong 1-\mu+\left(\pi-1\right)\left(u_t-\mu\right)+\psi, \\ p_t &\cong \mu-b+\left[1-\pi\left(1+\beta\right)\right]\left(u_t-\mu\right)+\psi_b, \\ q_t &\cong 1+\pi\left(u_t-\mu\right)+\psi_p, \end{aligned} \tag{29}$$

where $\psi \equiv \psi_p-\psi_k=O\left(h^2\right)$ and $\psi_b \equiv \psi_k-\left(1+\beta\right)\psi_p=O\left(h^2\right)$ since $\psi_k=$

$O\left(h^2\right)$ and $\psi_p = O\left(h^2\right)$. Equation (29) and the Conjecture imply

$$\begin{aligned}
\mathrm{E}_t\left[p_{t+1}\right] &\cong \mu - b + \psi_b = \mu - b + O\left(h\right), \qquad (30)\\
p_t \mathrm{E}_t\left[p_{t+1}\right]^{-1} &\cong 1 + \pi\left(u_t - \mu\right) \cong 1 + \frac{1 - \pi\left(1+\beta\right)}{\mu - b}\left(u_t - \mu\right) \Rightarrow \\
\pi &= \frac{1 - \pi\left(1+\beta\right)}{\mu - b} + O\left(h^2\right).
\end{aligned}$$

The final equation is consistent with $\pi = \left(\beta_0 + \mu\right)^{-1}$. This confirms the conjecture used for deriving equation (21). ■

Proposition 6

Substitution of the value for π in equation (29) yields equation (22). Statements 1-4 follow immediately.

Expectation and variance of q_t: Equation (22) implies the following:

$$\begin{aligned}
\mathrm{E}\left[q_t\right] &\cong 1 + \psi_p, \mathrm{Var}\left[q_t\right] \cong \mathrm{Var}\left[q_t^{-1}\right] \cong \pi^2\sigma^2,\\
\mathrm{E}\left[q_t^{-1}\right] &\cong 1 - \psi_p + \mathrm{Var}\left[q_t\right] \cong 1 - \psi_p + \pi^2\sigma^2,\\
\mathrm{Cov}\left[q_t, q_t^{-1}\right] &= \mathrm{E}\left[q_t q_t^{-1}\right] - \mathrm{E}\left[q_t\right]\mathrm{E}\left[q_t^{-1}\right] \cong -\pi^2\sigma^2,\\
1 - \eta z_t &\cong 1 + \eta b \psi_p + \eta b \pi\left(u_t - \mu\right) \Rightarrow\\
\mathrm{E}\left[1 - \eta z_t\right] &\cong 1 + \eta b \psi_p, \mathrm{Var}\left[1 - \eta z_t\right] \cong \eta^2 b^2 \pi^2 \sigma^2.
\end{aligned}$$

Expectation and variance of R_t: Substituting equation (28) in equation (25) and using $f_{kt} = \mathrm{E}_t\left[f_{kt}\right]\left(1 + v_{t+1}\right)$ and $p_{t+1} \cong \mathrm{E}_{t+1}\left[p_{t+2}\right]\left[1 + \pi\left(u_{t+1} - \mu\right)\right]$ (see equation (18) and the Conjecture) yields:

$$\begin{aligned}
R_t &\cong \left(1 - s_t - g_t\right) q_t^{-1} + s_t f_{kt} + g_t p_t^{-1} p_{t+1} - 1\\
&\cong \left(1 - s_t - g_t\right) q_t^{-1} + s_t\left(q_t^{-1} + \psi_k\right)\left(1 + v_{t+1}\right)\\
&\quad + g_t\left(q_t^{-1} + \psi_b\right)\left[1 + \pi\left(u_{t+1} - \mu\right)\right] - 1,\\
&\cong q_t^{-1} - 1 + \left[g_t \pi\left(u_{t+1} - \mu\right) + s_t v_{t+1}\right] q_t^{-1} + \Psi,\\
\Psi &\equiv \left(1 - \mu\right)\psi_k + \left(\mu - b\right)\psi_p = O\left(h^2\right),
\end{aligned}$$

using $\psi_p = O\left(h^2\right)$ and $\psi_k = O\left(h^2\right)$ in the third line. Hence

$$\begin{aligned}
\mathrm{E}\left[R_t\right] &\cong \mathrm{E}\left[q_t^{-1}\right] - 1 + \Psi \cong -\psi_p + \pi^2\sigma^2 + \Psi = O\left(h^2\right),\\
\mathrm{Var}\left[R_t\right] &\cong \mathrm{Var}\left[q_t^{-1}\right] + \mathrm{Var}\left[g_t \pi\left(u_{t+1} - \mu\right) + s_t v_{t+1}\right] \cong \pi^2\sigma^2 + \gamma^{-1}\Psi,
\end{aligned}$$

using $\mathrm{E}[g_t\pi\left(u_{t+1}-\mu\right)+s_tv_{t+1}]=0$ in the first line and $\mathrm{Var}[g_t\pi\left(u_{t+1}-\mu\right)+s_tv_{t+1}]=O\left(h^2\right)$ and $q_t^{-1}=1+O\left(h\right)$ in the second line.

Expectation and variance of c_t^y and c_{t+1}: Consumption of the young and the elderly satisfies the following

$$\begin{aligned}
\eta_0 c_t^y &= 1-\eta z_t \cong 1+\eta b\psi_p+\eta b\pi\left(u_t-\mu\right), \\
c_{t+1} &= \left(1-\eta z_t\right)\left(R_t+1\right).
\end{aligned}$$

Using the results on the expectation and variance of q_t and R_t yields

$$\begin{aligned}
\mathrm{E}\left[\eta_0 c_t^y\right] &\cong 1+\eta b\psi_p, \\
\mathrm{Var}\left[\eta_0 c_t^y\right] &\cong \eta^2b^2\pi^2\sigma^2, \\
\mathrm{E}\left[c_{t+1}\right] &\cong 1+\eta b\psi_p+\mathrm{E}\left[R_t\right]+\eta b\mathrm{Cov}\left[q_t,q_t^{-1}\right]\cong 1-\beta_0\left(\psi_p-\pi^2\sigma^2\right)+\Psi, \\
\mathrm{Var}\left[c_{t+1}\right] &\cong \eta^2b^2\mathrm{Var}\left[q_t\right]+\mathrm{Var}\left[R_t\right]+2\eta b\mathrm{Cov}\left[q_t,q_t^{-1}\right]\cong \eta b\left(1-2\eta b\right)\pi^2\sigma^2+\gamma^{-1}\Psi.
\end{aligned}$$

By Lemma 8 and using the expression above, the certainty equivalents of consumption of the young and the elderly satisfy

$$\begin{aligned}
\mathrm{E}\left[\left(\eta_0 c_t^y\right)^{1-\gamma}\right]^{1/(1-\gamma)} &\cong 1+\eta b\left(1-\frac{1}{2}\eta b\right)\psi_p=1+O\left(h^2\right), \\
\mathrm{E}\left[c_{t+1}^{1-\gamma}\right]^{1/(1-\gamma)} &\cong 1-\beta_0\left(\psi_p-\pi^2\sigma^2\right)+\frac{1}{2}\eta b\left(1-2\eta b\right)\gamma\pi^2\sigma^2+\frac{1}{2}\Psi=1+O\left(h^2\right).
\end{aligned}$$

Equation (17) can be written as

$$\eta^{-1}U=\eta_0^{-1}\mathrm{E}\left[\left(\eta_0 c_t^y\right)^{1-\gamma}\right]^{1/(1-\gamma)}+\mathrm{E}\left[c_{t+1}^{1-\gamma}\right]^{1/(1-\gamma)}-\eta_0^{-1}\ln\eta_0,$$

since $\mathrm{E}\Big[\left(\eta_0 c_t^y\right)^{1-\gamma}\Big]^{1/(1-\gamma)}-1$ and $\mathrm{E}\Big[c_t^{1-\gamma}\Big]^{1/(1-\gamma)}-1$ are of order $O\left(h^2\right)$.

Hence, the first-order condition for the optimal value of b reads:[7]

$$0\cong\eta_0^{-1}\frac{d\mathrm{E}\left[\left(\eta_0 c_t^y\right)^{1-\gamma}\right]^{1/(1-\gamma)}}{db}+\frac{\mathrm{E}\left[c_{t+1}^{1-\gamma}\right]^{1/(1-\gamma)}}{db}.$$

Hence, the sign of $dU/db_{|b=0}$ up to an order $O\left(h^3\right)$ is equal to the sign of

$$\left(1-\mu\right)\eta+\gamma\left(1+\frac{1}{2}\eta+\mu-\frac{3}{2}\mu\eta\right)$$

[7] Can be eliminated later

using $d\Psi/db = -2\psi_p$. The second term measures the effect of risk aversion. The most unfavorable case is $\mu = \eta = 1$. Even in that case, the second term is positive. This proves statement 7.■

Lemma 9

$$
\begin{aligned}
& \frac{d}{db}\left[\eta_0^{-1}\left(1+\eta b\left(1-\frac{1}{2}\eta b\right)\psi_p\right)+1-\beta_0\left(\psi_p-\pi^2\sigma^2\right)+\frac{1}{2}\eta b\left(1-2\eta b\right)\gamma\pi^2\sigma^2+\frac{1}{2}\Psi\right] \\
= & \frac{d}{db}\left[\left(\gamma\left(1-\eta\right)b\left(1-\frac{1}{2}\eta b\right)\left(\mu-b\right)-\gamma\beta_0\left(\mu-b\right)+\beta_0+\frac{1}{2}\eta b\left(1-2\eta b\right)\gamma\right)\pi^2\sigma^2\right]-\psi_p \\
= & \left[\gamma\left(1-\eta\right)\left(1-\eta b\right)\mu-\gamma\left(1-\eta\right)b\left(2-\frac{3}{2}\eta b\right)+\gamma\left(1-2\eta b\right)+\gamma\eta\mu-\eta+\frac{1}{2}\eta\left(1-4\eta b\right)\gamma\right]\pi^2\sigma^2+ \\
& 2\left[\left(\gamma\left(1-\eta\right)b\left(1-\frac{1}{2}\eta b\right)\left(\mu-b\right)-\gamma\beta_0\left(\mu-b\right)+\beta_0+\frac{1}{2}\eta b\left(1-2\eta b\right)\gamma\right)\right]\pi^3\sigma^2\eta-\gamma\left(\mu-b\right)\pi^2\sigma^2
\end{aligned}
$$

For $b = 0$ and dividing by $\pi\sigma^2$

$$
\left(1-\mu\right)\eta+\gamma\left(1+\frac{1}{2}\eta+\mu-\frac{3}{2}\mu\eta\right)
$$

$b = 0, \beta_0 = 1$

$$
\begin{aligned}
= & \left(\begin{array}{c}
\gamma\left(1-\eta\right)\left(1-\eta b\right)\mu-\gamma\left(1-\eta\right)b\left(2-\frac{3}{2}\eta b\right)+\gamma\left(1-2\eta b\right)+\gamma\eta\mu-\eta+\frac{1}{2}\eta\left(1-4\eta b\right)\gamma+ \\
2\left(\left(\gamma\left(1-\eta\right)b\left(1-\frac{1}{2}\eta b\right)\left(\mu-b\right)-\gamma\beta_0\left(\mu-b\right)+\beta_0+\frac{1}{2}\eta b\left(1-2\eta b\right)\gamma\right)\right)\pi\eta-\gamma\left(\mu-b\right)
\end{array}\right) \\
= & \gamma-\eta+2\left(1+\mu\right)^{-1}\eta+\frac{1}{2}\gamma\eta-2\left(1+\mu\right)^{-1}\gamma\mu\eta \\
= & \left(1+\mu\right)^{-1}\left(\left(\gamma-\eta+\frac{1}{2}\gamma\eta\right)\left(1+\mu\right)+2\left(1-\gamma\mu\right)\eta\right)=\left(1+\mu\right)^{-1}\left(\gamma+\frac{1}{2}\gamma\eta+\left(\gamma-\eta+\frac{1}{2}\gamma\eta\right)\mu+\left(1-2\gamma\mu\right)\eta\right) \\
= & \left(1+\mu\right)^{-1}\left(\left(1-\mu\right)\eta+\gamma\left(1+\frac{1}{2}\eta+\mu-\frac{3}{2}\mu\eta\right)\right)
\end{aligned}
$$